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THE COMMUTING FUNCTION CONJECTURE

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "THE COMMUTING FUNCTION CONJECTURE", submitted by JOHN A. HORNBY in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

In 1954 E. Dyer conjectured that any pair of continuous functions mapping the unit interval into itself and which commute under composition must have a common fixed point. This thesis is a survey of the work dealing with Dyer's conjecture. Embodied in the thesis are a number of special cases for which the conjecture is true and two distinct classes of counter examples.

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CHAPTER I

THE COMMUTING FUNCTION CONJECTURE

Let f and g be continuous functions mapping the unit interval $I = [0,1]$ into itself which commute in the sense that $f(g(x)) = g(f(x))$ for all x in I. According to Baxter [2] the conjecture that f and g must have a common fixed point was first made by E. Dyer in 1954. Both Baxter and Boyce [8] claim that later, in 1955, A. L. Shields made the same conjecture as did L. Dubins. The question appeared as part of a more general problem posed by J. R. Isbell [18] in 1957.

From 1954, when Dyer introduced the problem, until 1963 no results were published concerning its solution. However in 1951 Block and Thielman [6] had observed by referring to Ritt's [23] work that if two non-linear polynomials commute with each other, then both polynomials are iterates of some polynomial or are of the form $\lambda^{-1} \circ p_n \circ \lambda$ with $p_n(x) = x^n$, or are of the form $\lambda^{-1} \circ T_n \circ \lambda$ where T_n is the n^{th} degree Tchebychev polynomial and λ is linear. It is clear that in the first two cases the conjecture is verified. As for the case where the two functions are of the form $\lambda^{-1} \circ T \circ \lambda$ we have that $\lambda^{-1} \circ T \circ \lambda$ maps some interval $[a,b]$ into itself since T maps $[-1,1]$ into itself. So for any homeomorphism $\varphi : [a,b] \to [0,1]$ $\varphi^{-1} \circ \lambda^{-1} \circ T_n \circ \lambda \circ \varphi$ maps $[0,1]$ onto itself. Since Tchebychev polynomials have a common fixed point the conjecture is verified for polynomials.

Then in 1963 Baxter and Joichi [3] and DeMarr [11] published positive results for special cases. Ritt's result led Baxter and Joichi

to try to embed the commuting functions in a one parameter semi-group so as to prove the existence of a common fixed point. They derived a necessary condition under which it is possible to embed f and g in what they called a "nice" semi-group of commuting continuous functions. Then they showed that the Tchebychev polynomials cannot be embedded in such a semi-group.

We will use the notation fg for $f \circ g$ from now on since it is convenient and cannot be confused with the pointwise product $f(x)\ g(x)$ because this does not appear. The n^{th} iterate of f will be denoted f^n. Again, this does not mean $[f(x)]^n$.

Suppose f and g commute and x is a fixed point of $h = gf = fg$ then $hg(x) = (gf)g(x) = g(fg)(x) = gh(x) = g(x)$. So $g(x)$ is also a fixed point of h. Furthermore if x is a fixed point of f as well as of h, then $g(x) = gf(x) = h(x) = x$; so x is a fixed point of g. Thus a study of the fixed points of h has merit. We shall consider a paper by Baxter [2] in which he studies the effect of f and g on the fixed points of the composite function h and another in which he and Joichi [3] extend this study.

In 1964 A. L. Shields [25] showed that a commuting family of functions which are holomorphic on the open unit disc of the complex plane and which map the closed unit disc D continuously into itself has a fixed point z_o common to all the functions in the family. This added support to the conjecture. In 1965 more indirect support for the conjecture was given by A. J. Schwartz [24]. He showed that if f is assumed to have a continuous derivative then f and some iterate of

g have a common fixed point (even if g has a infinite number of fixed points). For then the conjecture reduces to the assertion that g is the particular iterate having the fixed point in common with f .

By the end of 1965 a number of papers had been published containing affirmative results for special cases. De Marr [11] found necessary conditions under which the functions f and g satisfying Lipschitz conditions would have a common fixed point. G. Jungck's paper [20] contains results related to De Marr's. He showed that if f and g commute on the unit interval and if there is a real number $\alpha > 0$ such that $|f(x)-f(y)| \leq \alpha|gf(x)-gf(y)| + |x-y|$ for all x,y in I then f and g have a common fixed point.

In 1964 H. Cohen [10] noticed that many of the known examples of commuting functions have a property which he called fullness. A function f is called a full function if there is a partition $\{0 = x_o, x_1, \ldots, x_n = 1\}$ of I such that for each interval $I_i = [x_i, x_{i+1}]$ $f|I_i$ is a homeomorphism onto I . Cohen shows that commuting full functions have a common fixed point. Cohen's full functions can be specialized to regular full functions in which the subintervals of the partition are all of the same length; and hat functions which are regular full functions for which the function is linear on each subinterval.

Baxter and Joichi [4] show that if one of the commuting pair of functions, say f , is a hat function then the other is either identically a constant c with c a fixed point of f or it is also a hat function. However a hat function is a full function. So by Cohen's result the two functions have a common fixed point. Continuing this

work Baxter and Joichi show that if one of the functions is a regular full function the conjecture is true.

In [19] Joichi verifies the conjecture whenever one of the functions is a full function. Independently and by a different method J. H. Folkman [12] obtains the same result.

Working in another direction Chu and Moyer [9] consider the set of functions which have the property that $f(x) \neq x$ implies that $f^2(x) \neq x$ for all x in a closed subset C of I. They show that if there is a closed subinterval of I such that $f(C)$ is contained in C, $f(x) \neq x$ implies $f^2(x) \neq x$ for every x in C, and g (the function which commutes with f) has a common fixed point in C, then f and g have a common fixed point. They also show that on a closed interval a number of other properties are equivalent to the property $f(x) \neq x$ implies $f^2(x) \neq x$.

Although special cases of the conjecture were verified, finally in 1967 Boyce [7] and Huneke [17] independently announced the existence of counter examples. One of Huneke's is essentially the same as Boyce's counter example. We will consider two counter examples. The first was constructed by Huneke only. He specifies a number of conditions which a pair of functions must satisfy in order that they be continuous, commute and have no fixed point. Then he proves that such a pair does exist. It is interesting to note that the functions f and g of Huneke's counter example have derivatives $\pm s$ a.e. on I, with $s = 3 + \sqrt{6}$.

The second counter example is obtained by constructing two uniformly convergent sequences of functions $\{f_n\}$ and $\{g_n\}$ such that for each n f_n and g_n are continuous, $f_n g_{n+1} = g_n f_{n+1}$, and the fixed points of f_n are bounded away from the fixed points of g_n. This counter example was arrived at independently by Boyce and Huneke. The functions do not have bounded derivatives if the derivatives exist at all.

CHAPTER II

POLYNOMIALS

In the early 1920's Ritt [23] in his work with composite polynomials determined necessary and sufficent conditions for a polynomial to be a composite of two or more polynomials. Ritt's method of attack uses deep results in the theory of complex variables. His proofs are long and involved. For this reason, we will limit our consideration of Ritt's work to a discussion of his work and its consequences.

We will call a polynomial which cannot be written as the composite of two or more polynomials a <u>prime</u> polynomial. Also if the polynomial "h" has two decompositions in terms of prime polynomials, say,

$$h = \alpha_1\alpha_2\ldots\alpha_n = \beta_1\beta_2\ldots\beta_n$$

we will say that these two decompositions are <u>equivalent</u> if there are (n-1) polynomials of the first degree, say $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ such that

$$\alpha_1 = \beta_1\lambda_1 \ , \quad \alpha_2 = \lambda_1^{-1}\beta_2\lambda_2 \ , \quad \ldots \quad \alpha_n = \lambda_{n-1}^{-1}\beta_n \ .$$

Suppose for a decomposition of h into prime polynomials say

$$h = \alpha_1\ \alpha_2\ \ldots\ \alpha_n \tag{1}$$

there is an adjacent pair of polynomials α_i and α_{i+1} such that

$$\alpha_i = \lambda_1 \pi_1 \lambda_2 \quad \text{and} \quad \alpha_{i+1} = \lambda_2^{-1} \pi_2 \lambda_3$$

where λ_i $(i = 1,2,3)$ are linear. Suppose also that the polynomials π_1 and π_2 are of degrees m and n respectively and that they are given by one of the following sets of equations.

(a) $\pi_1(z) = T_m(z)$ $\pi_2(z) = T_n(z)$; $T_n(\cos(z)) = \cos nz$

(b) $\pi_1(z) = z^m$ $\pi_2(z) = z^r \cdot g(z^m)$; g is any polynomial

(c) $\pi_1(z) = z^r \cdot [g(z)]^m$ $\pi_2(z) = z^n$; $[g(z)]^m$ is the m^{th} power of $g(z)$

Then we can write for h a decomposition distinct from 1 . Namely

$$h = \alpha_1 \alpha_2 \ldots \alpha_{i-1} \beta_1 \beta_2 \alpha_{i+2} \ldots \alpha_n \tag{2}$$

where $\beta_1 = \lambda_1 \pi_1' \lambda_2$ and $\beta_2 = \lambda_2^{-1} \pi_2' \lambda_3$ and

$\pi_1' = \pi_2$ $\pi_2' = \pi_1$ when π_1 and π_2 are as in (a)

$\pi_1' = \pi_1$ of (c) $\pi_2' = \pi_2$ of (c) when π_1 and π_2 are as in (c)

$\pi_1' = \pi_1$ of (b) $\pi_2' = \pi_2$ of (b) when π_1 and π_2 are as in (b)

A T-transformation is a transformation which changes a decomposition (1) into a decomposition (2) as above.

Ritt shows

(I) If h has two distinct decompositions into prime polynomials it is possible to transform either decomposition into a decompsotion equivalent to the other by a finite numoer of T-transformations.

In order to obtain this result Ritt first shows that any two decompositions of a given polynomial into prime polynomials contain the same number of polynomials. Furthermore the degrees of the polynomials in one decomposition are the same as those in the other decomposition. This result, along with most of the others, follows from a study of the branches of the algebraic function h^{-1} in the neighborhoods of critical points of h^{-1} and α_i^{-1} $(i = 1,2,\ldots,n)$.

When (I) has been obtained the problem is reduced from finding the necessary and sufficient conditions for h to be the composite of prime polynomials to answering the following question: what are the necessary and sufficient conditions under which $\varphi\alpha = \psi\beta$ where φ and β are prime polynomials of degree $m > 1$ and α and ψ are prime polynomials of degree $n > 1$?

After showing that this problem is equivalent to the same question but with the added restriction that m and n be relatively prime, Ritt shows that one of ψ^{-1} and φ^{-1} cannot have more than two critical points in addition to infinity. Suppose it is ψ^{-1} .

If ψ^{-1} has only one critical point Ritt shows by a consideration of the orders of the critical points of α^{-1} and ψ^{-1} that

$$\begin{aligned} \varphi &= \lambda_1 \varphi_1 \lambda_2 & \qquad \alpha &= \lambda_2^{-1} \alpha_1 \lambda_3 \\ \psi &= \lambda_1 \psi_1 \lambda_4 & \qquad \beta &= \lambda_4 \beta_1 \lambda_3 \end{aligned} \tag{a}$$

where λ_i $(i = 1,2,3,4)$ are linear and where

$$\varphi_1(z) = z^r[g(z)]^n \qquad \alpha_1(z) = z^n$$

$$\psi_1(z) = z^n \qquad \beta_1(z) = z^r[g(z^n)]$$

If ψ^{-1} has two critical points, then

$$\varphi = \lambda_1 T_m \lambda_3 \qquad \alpha = \lambda_3^{-1} T_n \lambda_2 \qquad T_n(\cos x) = \cos nx$$

$$\psi = \lambda_1 T_n \lambda_4 \qquad \beta = \lambda_4^{-1} T_m \lambda_2 \quad . \tag{b}$$

If ψ^{-1} has one critical point we will say that $\varphi, \psi, \alpha,$ and β are type (a) . If ψ^{-1} has two critical points we will say that $\varphi, \psi, \alpha,$ and β are of type (b) . Thus Ritt showed that all of $\varphi, \psi, \alpha,$ and β are of one type either (a) or (b) .

<u>Definition</u>: If $\lambda(x) = Ax+B$ $(A \neq 0)$, then $\lambda^{-1} f \lambda$ is called the <u>linear transformation of</u> f <u>by</u> λ .

Let f and g be polynomials such that

$$h = fg = gf \tag{2}$$

Suppose

$$f = \alpha_1\alpha_2 \ldots \alpha_r$$

$$g = \beta_1\beta_2 \ldots \beta_s$$

are prime decompositions for f and g . Then (2) becomes

$$\alpha_1\alpha_2 \ldots \alpha_r\beta_1\beta_2 \ldots \beta_s = \beta_1\beta_2 \ldots \beta_s\alpha_1\alpha_2 \ldots \alpha_r \quad . \tag{3}$$

We may assume without loss of generality that $s \geq r$. It is clear that if all the α_i and β_j are of type (a) then f and g are linear transforms of powers of x and if all the α_i and β_j are of type (b) then f and g are linear transforms of the Tchebychev polynomials.

The case where the α_i and the β_j are not all of the same type is somewhat more complex. From (I) we know it is possible to transform the decomposition of the L.H.S. (R.H.S.) of (3) into one equivalent to the R.H.S. of (3) in a finite number of moves. Since not all the factors of h are of the same type there must be at least one factor, say α_i of type (b) . Thus at least one of α_{i-1} or α_{i+1} is of type (b) unless $\beta_i = \alpha_i$. Now T-transformations preserve type so β_i and at least one of β_{i-1} and β_{i+1} are of type (b) . Furthermore, since the T-transformations can only be applied to two adjacent factors of the same type we can partition the factor of f and g into blocks of type (a) and type (b).

We will say that α_1 has type (a') and that the other type is type (b'). It is clear, then, that α_2 is also of type (a') unless β_1 and α_1 are the same. For if β_1 is not the same as α_1 and α_2 is of type (b') then the two decompositions could not be transformed into a pair of equivalent decompositions contrary to (I) . Thus, if the first $n \leq r$ factors of f are of type (a'), so are the first n factors of g , and from the last sentence of the last paragraph we have

$$\alpha_1\alpha_2\ldots\alpha_n = \beta_1\beta_2\ldots\beta_n \quad .$$

Similarly if α_i is of type (b') for $n < i \leq m \leq r$ we have

$$\alpha_{n+1}\alpha_{n+2}\cdots\alpha_m = \beta_{n+1}\beta_{n+2}\cdots\beta_m \quad .$$

Carrying on in this fashion we get;

$$\begin{array}{cccccc}
\underbrace{\alpha_1\alpha_2\cdots\alpha_n} & \underbrace{\alpha_{n+1}\cdots\alpha_m} & \underbrace{\alpha_{m+1}\cdots\alpha_r} & \underbrace{\beta_1\beta_2\cdots\beta_n} & \beta_{n+1}\cdots\beta_m\cdots\underbrace{\beta_{s-r}} & \underbrace{\beta_{s-r+1}\cdots\beta_s} \\
\| & \| & \| & \| & \| & \| \\
\overbrace{\beta_1\beta_2\cdots\beta_n} & \overbrace{\beta_{n+1}\cdots\beta_m} & \overbrace{\beta_{m+1}\cdots\beta_r} & \overbrace{\beta_{r+1}\beta_{r+2}\cdots} & \overbrace{\cdots\beta_s} & \overbrace{\alpha_1\cdots\alpha_r}
\end{array}$$

and we see that

$$\beta_1\beta_2\cdots\beta_n = \beta_{k+1}\beta_{k+2}\cdots\beta_{k+n} \quad \text{for every } k < s,\ k \equiv 0 (\mathrm{mod}\ r)$$

similarly

$$\beta_{n+1}\beta_{n+2}\cdots\beta_m = \beta_{k+n+1}\beta_{k+n+2}\cdots\beta_{k+m} \quad \text{for every } k < s,\ k \equiv 0 (\mathrm{mod}\ r)$$

and we see that either f and g are iterates of some polynomial or g is an iterate of f .

An entire set of commutative polynomials is a set of polynomials which contains at least one of each positive degree and is such that each pair of polynomials in the set commute under composition. A result, weaker than Ritt's, was given by Block and Thielman [6] in 1951. They proved the following theorem.

<u>Theorem 2.1</u> The only entire sets of commutative polynomials are the sets

(I) $\{P_n | n = 1,2,\ldots\}$ where $P_n = \lambda^{-1}p_n\lambda$, $p_n(x) = x^n$ and λ is linear

(II) $\{T_n | n = 1,2,\ldots\}$ where $T_n(x) = \lambda^{-1} \cos n(\arc \cos \lambda(x))$

Block and Thielman use more elementary methods to attain their results than Ritt did to arrive at his. In essence they show that if a second degree polynomial, say h_2, commutes with a third degree polynomial h_3 then the second degree polynomial must be one of two types. Now every entire set of commutative polynomials contains a second degree polynomial and a third degree polynomial. Thus in a given entire set of commutative polynomials h_2 has one of two possible forms.

The one form leads to the conclusion that $h_2 = P_2$. That $h_3 = P_3$ and $h_n = P_n$ follows from a consideration of the zeros of the composite functions.

The other form gives $h_2 = T_2$. Now $T_2 h_n = h_n T_2$. By differentiating both sides Block and Thielman arrive at a differential equation which h_n must satisfy. Since the only solution to the equation is T_n, the theorem is proved. We shall now give details of the argument.

<u>Lemma 2.2</u> Let h_2 be a second degree polynomial namely $h_2(x) = \alpha x^2 + \beta x + \gamma$ and let $\Delta = \beta^2 - 4\alpha\gamma$. Let h_3 be a third degree polynomial.

If $h_2 h_3(x) = h_3 h_2(x)$ for $x \in C$, then

$$\Delta = 2\beta \qquad \text{or} \qquad \Delta = 2\beta + 8 \ .$$

<u>Proof</u>. Let $\varphi(x) = \alpha x + \beta/2 = y$, and $g_k(y) = \varphi h_k \varphi^{-1}(y)$ where $k = 2,3$.

Then $g_2 g_3(y) = g_3 g_2(y)$. Furthermore

$$\alpha h_2(x) = \alpha(\alpha x^2 + \beta x + \gamma) = (\alpha x + \beta/2)^2 - \frac{1}{4}(\beta^2 - 4\alpha\gamma) = y^2 - \frac{\Delta}{4}$$

$$\therefore \quad g_2(y) = \varphi h_2 \varphi^{-1}(\varphi(x)) = \varphi h_2(x) = \alpha h_2(x) + \beta/2 = y^2 - \delta \,, \quad \text{where} \quad \delta = \frac{\Delta}{4} - \frac{\beta}{2} \,.$$

Let

$$g_3(y) = Ay^3 + By^2 + Cy + D \quad .$$

Then

$$g_3 g_2(y) = A(y^2 - \delta)^3 + B(y^2 - \delta)^2 + C(y^2 - \delta) + D$$

and

$$g_2 g_3(y) = (Ay^3 + By^2 + Cy + D)^2 - \delta \quad .$$

Equating corresponding coefficients

$$\begin{array}{lll}
A = A^2 & (y^6) & \text{so} \quad A = 1 \\
0 = 2AB & (y^5) & \text{so} \quad B = 0 \\
-3\delta = B^2 + 2AC & (y^4) & \text{so} \quad C = -\frac{3}{2}\delta \\
0 = 2D & (y^3) & \text{so} \quad D = 0 \\
3\delta^2 - \frac{3}{2}\delta = \frac{4}{4}\delta^2 & (y^2) & \text{so} \quad \delta^2 - 2\delta = 0 \quad \text{and} \quad \delta = 0 \quad \text{or} \quad 2 \,.
\end{array}$$

However

$$\Delta = 4\delta + 2\beta$$

so

$$\Delta = 2\beta \quad \text{or} \quad 2\beta + 8 \,.$$

<u>Proof (of Theorem 2.1)</u> If $h_2 = \lambda^{-1} p_2 \lambda$ for some λ then

$$h_2(x) = \frac{(Ax+B)^2 - B}{A} = Ax^2 + 2Bx + \frac{B(B-1)}{A}$$

so $\triangle = 2\beta$ where $\beta = 2B$. If $\triangle = 2\beta$ let $A = \alpha$ and $B = 2\beta$ then $h_2 = \lambda^{-1} p_2 \lambda$ where $\lambda = Ax + B$. Similarly $\triangle = 2\beta + 8$ if and only if there is a λ such that $h_2(x) = \lambda^{-1} \cos 2(\arc \cos \lambda(x))$.

Let the set $\{h_k\}$ be an entire set of commutative polynomials, then h_3 commutes with h_2 so $\triangle = 2\beta$ or $2\beta+8$ (by lemma 2). Thus by the first paragraph of this proof, h_2 is P_2 or T_2 .

Suppose $h_2 = P_2$ and h_k commutes with h_2 . Then $H_k = \lambda h_k \lambda^{-1}$ commutes with x^2 . Thus

$$H_k(x^{2n}) = [H_k(x)]^{2n} \quad \text{where} \quad n \in I^+ \tag{4}$$

Equation (4) shows that every zero of $H_k(x^{2n})$ is also a zero of $H_k(x)$. Thus if $x_o = re^{i\theta}$ $(r \neq 0, \theta \neq 0)$ is a zero of $H_k(x)$ so is $r^{\frac{1}{2n}} e^{\frac{i\theta}{2n}}$. This gives an infinite number of distinct zeros for $H_k(x)$, which is impossible. Thus every zero of $H_k(x)$ is the number 0 and $H_k(x) = ax^k$. However equation (4) shows that $a = 1$. Since $H_k = \lambda h_k \lambda^{-1}$ we have

$$h_k = \lambda^{-1} H_k \lambda = \lambda^{-1} x^k \lambda = P_k \quad .$$

If h_k commutes with T_2 then $H_k = \lambda h_k \lambda^{-1}$ commutes with the function h defined by $h(x) = 2x^2-1$ so

$$H_k(2x^2 - 1) = 2[H_k(x)]^2 - 1 \quad . \tag{5}$$

Differentiating both sides of (5) gives

$$x\, H_k'(2x^2-1) = H_k(x)H_k'(x) \tag{6}$$

Now let

$$F(x) = (1-x^2)\cdot[H_k'(x)]^2 + k^2\{[H_k(x)]^2-1\} \quad . \tag{7}$$

Then

$$\begin{aligned}
F(2x^2-1) &= [1-(2x^2-1)^2]\cdot[H_k'(2x^2-1)]^2 + k^2\{[H_k(2x^2-1)]^2-1\} \\
&= [1-(2x^2-1)^2]\left[\frac{H_k(x)\cdot H_k'(x)}{x}\right]^2 + k^2\{[2\{H_k(x)\}^2-1]^2-1\} \\
&= [1-(4x^4-4x^2-1)][H_k(x)]^2[H_k'(x)]^2 + k^2\{4[H_k(x)]^4 - 4[H_k(x)]^2+1-1\} \\
&= [4-4x^2][H_k(x)]^2[H_k'(x)]^2 + 4k^2\{[H_k(x)]^4 - [H_k(x)]^2\} \\
&= 4[H_k(x)]^2\{(1-x^2)[H_k'(x)]^2 - k^2([H_k(x)]^2 - 1)\} \\
&= 4[H_k(x)]^2\, F(x) \quad .
\end{aligned}$$

i.e. $$F(2x^2 - 1) = 4[H_k(x)]^2\, F(x) \tag{8}$$

Clearly $F(x)$ is of a degree less than $2k$. But if $F(x) \not\equiv 0$ then equation (8) shows that the degree of $F(x)$ is $2k$. Thus $F(x) \equiv 0$ and (7) becomes a differential equation in $H_k(x)$. From (5) we have $H_k(1) = 1$. The only polynomial solution of the differential equation satisfying this condition and (5) is

$$H_k(x) = \cos k(\arccos x)$$

thus $h_k = T_k$. Hence the theorem is proved.

CHAPTER III

HOLOMORPHIC FUNCTIONS ON THE UNIT DISC

In this chapter we shall consider a paper by Shields [25] which contains results closely related to the commuting function conjecture. In his paper Shields considers the fixed point conjecture for commuting families of holomorphic functions mapping the unit disc into itself. Shields proves

<u>Theorem 3.1</u> (Shields [25]) Let F be a commuting family of continuous functions each mapping the closed unit disc into itself and holomorphic in the open unit disc D. Then there is a common fixed point for all the functions of the family.

Shields shows that if f is holomorphic in the unit disc into itself and if f is not a homeomorphism of the disc into itself then there is a point z_o in the disc and a sequence of integers $\{n_i\}$ for which $f^{n_i}(z)$ converges to z_o uniformly on compact subsets of D. This result is vital to the proof of the theorem and a good share of Shields paper is devoted to obtaining it.

Having reached that point Shields is then able to show that the point z_o must be a fixed point common to all functions which are continuous and commute with f. Thus the theorem is proved for families in which one of the members is not a homeomorphism of D into itself.

Therefore we must consider the case in which every member of the commuting family is a homeomorphism of D into itself. Shields

shows that if f is such a homeomorphism then either the function f has one or two fixed points, or it is the identity function. If the family contains a function with only one fixed point the result follows immediately.

Suppose all the elements in the family F have two fixed points. Also assume that the iterates of a function f in F tend to one of the fixed points of f. Then by adding the set $\{f^n\}$ of iterates of f, we obtain a commuting family containing F with a fixed point common to all the functions. Thus the functions of F must have a common fixed point. Thus the second aim is to show that if f is any homeomorphism of D onto itself with two fixed points then the iterates of f converge to one of these fixed points.

The next few lemmas are aimed at showing the existence of a point z_o and a sequence $\{n_i\}$ such that the iterates f^{n_i} of a given holomorphic homeomorphism f converge to z_o uniformly on compact subsets of D. We have seen that this result is used in the proof of theorem (3.1).

We will need some notation. Let F_D denote the set $F_D = \{f | f(D) \subset D$ and f is holomorphic in $D\}$. Then a metric on F_D which is equivalent to uniform convergence on compact subsets of D is defined as follows (see Goffman and Pedrick [13] p. 35).

Let $\{K_n\}$ be a sequence of compact subsets of D such that K_n is contained in K_{n+1} for each n and $D = \bigcup_{n=1}^{\infty} K_n$. If d_n is the uniform metric on K_n, let

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)} .$$

The main purpose of the next two lemmas is to show that (F_D, o), where o denotes composition, is a topological semi-group.

Lemma 3.2. (Shields [25]). Let f_n and g_n be in F_D for every n. If the sequence $\{f_n\}$ converges to f in the above metric and $\{g_n\}$ converges to g, then $\{f_n g_n\}$ converges to fg.

Proof. Let k be a compact subset of D, then $g(k)$, being the continuous image of a compact set is compact and hence closed. Since $g(k)$ is contained in the open set D, there is an open set U containing $g(k)$ such that the closure of U, $\overline{U}$ is contained in D. Letting U^c denote the complement of U we see that the intersection of $g(k)$ and U^c is empty. So the compactness of $g(k)$ implies that $\beta = \inf \{|x-y| \mid x \text{ in } g(k), y \text{ in } U^c\}$ is strictly greater than zero. Since $\{g_n\}$ converges to g uniformly on k, there is a positive integer, N, such that $\|g_n - g\| < \beta/2$ whenever $n > N$, i.e. $|g_n(z) - g(z)| < \beta/2$ for every z in k provided $n > N$. So for $n > N$ $g_n(z)$ is not in U^c. Hence $g_n(z)$ is in U for each z in k. Now

$$|fg(z) - f_n g_n(z)| \leq |fg(z) - fg_n(z)| + |fg_n(z) - f_n g_n(z)| \qquad (1)$$

Let $\epsilon > 0$ be given. Since g_n converges to g uniformly on k and f is uniformly continuous on the closure of U, there is a $\delta = \delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon/2$ whenever x, y are in U

and $|x-y| < \delta$ and also an $N = N(\delta)$ such that $|g_n(z)-g(z)| < \delta$ for all z in k whenever $n > N$. So $|fg(z)-fg_n(z)| < \epsilon/2$ for all z in k whenever $n > N$. Since f_n converges to f uniformly on the closure of U, there is an integer N_2 such that $|f_n(z)-f(z)| < \epsilon/2$ for all z in U whenever $n > N_2$. Thus for $n > \max\{N,N_2\}$ we have

$$|fg(z)-f_ng_n(z)| \leq \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for all } z \text{ in } k .$$

which was to be shown.

We will need to know the nature of the idempotent elements in F_D i.e. elements e such that $e^2(z) = e(z)$ for all z in D. The following lemma characterizes the idempotents of F_D

Lemma 3.3 (Shields [24]). If e in F_D is idempotent, then either $e(z) = z$ or $e(z)$ is identically a constant.

Proof. Let $D_1 = e(D)$ and z_o be an element of D_1. Consider the set $e^{-1}(z_o)$. Since $ee(z) = e(z)$ for all z in D we have $eee^{-1}(z_o) = ee^{-1}(z_o)$ and since $ee^{-1}(z_o) = z_o$, $e(z_o) = z_o$. Thus $e(z) \equiv z$ for every z in D_1. If $e(z)$ is not constant then by the local inverse theorem for analytic functions, (see Hille V. 1 [15], p. 87) D is open. Hence $e(z) = z$ on an open set. Therefore by the Identity theorem (Hille V. 1 [15], p. 199) $e(z) = z$ on all of D. Thus the lemma is proved.

The next lemma is a simple corollary to a more general result

proved by Katsumi Numakura [22]. Numakura's lemma is as follows:

Lemma: Let S be a compact topological semi-group and "a" an element of S and let $A = \{a^n | n = 1,2,\ldots\}$. Then $\bar{A}$ the closure of A contains a commutative group in D.

However lemma 3.4 will be sufficient for our needs. The proof of lemma 3.4 is embodied in the proof of Numakura's lemma.

Let f be in F_D and $\Gamma(f) = \{f_1, f_2, \ldots\}^-$ be the closure of the set of iterates of f in F_D. Then by Montel's theorem (Goffman and Pedrick [13], p. 36) $\Gamma(f)$ is compact since $f(D)$ is contained in D. Similarly $\Gamma_n(f) = \{f^k | k \geq n\}^-$ is compact.

Lemma 3.4 (Numakura [22]). The set $G = \bigcap_{n=1}^{\infty} \Gamma_n(f)$ is nonempty and is a group under the operation of composition.

Proof. $U = \{\Gamma_n(f) | n = 1,2,\ldots\}$ is a family of closed subsets of a compact set which possesses the finite intersection property. Therefore $G = \bigcap_{n=1}^{\infty} \Gamma_n(f)$ is nonempty.

Clearly G is a semi-group under composition. Thus to show that G is a group we must show that each element has an inverse. This can be done by showing that $gG = \{gf | f \text{ in } G\} = G$ for all g in G.

Suppose there is a g in G such that $gG \subsetneq G$, then there is an h in G such that h is not in gG i.e. $h \neq gf'$ for any f' in G. Now by lemma 3.2 we have that composition is a continuous

operation on $F_D \times F_D$ to F_D. Therefore there are neighbourhoods $V_{f'}(g)$ of g, $V(f')$ of f' and $V_{f'}(h)$ of h in the metric topology such that the intersection of $V_{f'}(h)$ and $V_f(g)V(f')$ is empty. Now $\{V(f') \mid \text{in } G\}$ is an open cover of G. However, G is a closed subset of a compact set and hence compact. Thus there is a finite subcover of G say $\{V(f_i') \mid i = 1,2,\ldots,k\}$.

Let $V(g)$ and $V(h)$ be neighbourhoods of g and h such that $V(g)$ is contained in $\bigcap_{i=1}^{\infty} V_{f_i'}(g)$ and $V(h)$ is contained in $\bigcap_{i=1}^{k} V_{f_i'}(h)$. Now let $Q = \bigcup_{i=1}^{k} V(f_i')$. Then Q is an open set containing G with the intersection of $V(h)$ and $V(g)Q$ being empty. But g is in $G = \bigcap_{n=1}^{\infty} \Gamma_n$, so there is an integer $m \geq 1$ such that f^m is in $V(g)$. Furthermore, since h is in G we have a sequence of integers n_i, with $n_i > m$ and $n_{i+1} > n_i$, for which f^{n_i} is in $V(h)$. Let $\ell_i = n_i - m$ and $\overline{A^{(\ell_i)}} = \{f^{\ell_j} \mid j = i,i+1,\ldots\}$. Then as above $T = \bigcap_{i=1}^{\infty} \overline{A^{(\ell_i)}}$ (where $\overline{A^{(\ell_i)}}$ is the closure in the metric topology) is not empty.

Evidently T is contained in G. Also, since Q is an open set containing G we have for any f_o in T a neighbourhood $V(f_o)$ of f_o contained in Q. Now f_o in T implies that there is an integer i_k for which f^{i_k} is in $V(f_o)$. Then f^{n_k} is in $V(h)$ and $f^{n_k} = f^{m+i_k} = f^m f^{i_k}$ is in $V(g)V(f_o)$. But $V(g)V(f_o)$ is contained in $V(g)Q$. Thus the intersection of $V(h)$ and $V(g)Q$ contains f^{n_k} and hence is non-empty, contradicting our assumptions. Thus $gG = G$. So G is a group.

Lemma 3.5 (Shields [25]). If the identity function is in $\Gamma(f)$, then every element of $\Gamma(f)$ has an inverse in $\Gamma(f)$.

Proof. Let G be as in lemma 3.4 and suppose e is the identity function. We show first that $e \in G$. That is e is in $\Gamma_n(f)$ for every n . If $e = f^k$ for some k then $\Gamma_n(f)$ is finite for all n and $\Gamma_n(f) = \Gamma_m(f)$ for all m and n . So e is in G . Otherwise e is a limit point of $\{f^k | k = 1,2,\ldots\}$. Hence e is a limit point for $\{f^k | k \geq n\}$ for each n . Thus e is in G under any circumstance.

Now we show that $g\Gamma(f)$ contains G for any g in $\Gamma(f)$. Let g be an element of $\Gamma(f)$. If $g = f^n$ for some n , then $g\Gamma(f) = \Gamma_{n+1}(f)$ which contains G . If g is a limit point of $\Gamma(f)$ then g is in $\Gamma_n(f)$ for every n so g is an element of G . Thus $gG = G$. Now G is contained in $\Gamma(f)$ so $g\Gamma(f)$ must contain G . Thus there is in $\Gamma(f)$ a g^{-1} such that $gg^{-1} = e$. Since g was an arbitrary element of $\Gamma(f)$ we have that every element in $\Gamma(f)$ has an inverse.

Lemma 3.6 (Shields [25]). If $\Gamma(f)$ contains a constant function, say e , then $\{f^n\}$ converges to e in the above metric.

Proof. If $e = f^{n_o}$ for some n_o then $e = f^n$ for all $n > n_o$ and we are done. Otherwise there is a sequence n_i , where $n_i < n_{i+1}$, such that f^{n_i} converges to e . To see that f^n converges to e , let $\epsilon > 0$ be given. Then there is a positive integer i such that $d(f^{n_i},e) < \epsilon$. Hence for $m > n_i$, say $m = n_i+\ell$

$$d(f^m,e) = d(f^{n_i+\ell},e) = d(f^{n_i}f^{\ell},e) < \epsilon$$

the inequality holding since $f^{\ell}(D)$ is contained in D . This completes the proof of the lemma.

Lemma 3.7 (Shields [25]) $\Gamma(f)$ contains exactly one idempotent element.

Proof. By lemma 3.3 the only idempotent elements in $\Gamma(f)$ are constant functions and the identity function. Clearly it is not possible to have two constant functions in $\Gamma(f)$, since all elements of $\Gamma(f)$ commute. Since the identity function is unique, the only possibility for two idempotent functions is for one to be a constant function and the other to be the identity function. However, by lemma 3.6, the sequence $\{f^n\}$ converges to the constant function and therefore no subsequence could converge to the identity function. Hence there is at most one idempotent element in $\Gamma(f)$.

The existence of an idempotent in $\Gamma(f)$ follows from lemma 3.4. Since G is contained in $\Gamma(f)$ and G is a group under composition, G has a unity element (which is unique). This element is idempotent and as we have seen the only idempotent element in $\Gamma(f)$.

Lemma 3.8 (Shields [25]) Let f be in F_D and suppose f is not a homeomorphism of D onto itself. Then there is a point z_o in the closure of D and a sequence $\{n_i\}$ of positive integers such that $f^{n_i}(z)$ converges to z_o uniformly on compact subsets of D .

Proof. Let $\Gamma(f)$ be as above. Then if $\Gamma(f)$ is contained in F_D we have from lemma 3.7 that $\Gamma(f)$ contains an idempotent function $e(z)$. But f is not a homeomorphism so it does not have an inverse. Thus by lemma 3.5 $e(z) = z$ is impossible. Therefore $e(z) = z_o$ with z_o in D. So by lemma 3.6 f^n converges to z_o.

If $\Gamma(f)$ is not contained in F_D, then there is a function g in $\Gamma(f)$ not in F_D. Now g is a limit of functions which map D into D so $g(D)$ is contained in the closure of D. But g is not in F_D. Hence there is a point z' in D such that $g(z')$ is not in D. Then $g(z) \equiv z_o$. For suppose $g(z) \not\equiv z_o$ and let $h(z) = g(z) - z_o$. Then $h(z) \not\equiv 0$ and has a zero z'. If now $f'_{n_i} = f^{n_i} - z_o$, then f'_{n_i} converges to h. Let U be any neighbourhood of z'. Then by Hurwitz's theorem (see Alfors [1], p. 174) we have $f'_{n_i}(z) = 0$ for some n_i where z is in U. So $f^{n_i}(z) = z_o$. But this contradicts the fact that the iterates of f map D into itself. Thus $g(z) \equiv z_o$ and the lemma is proved.

We are now in a position to prove the following important lemma.

Lemma 3.9 (Shields [25]) Let f be in F_D and assume that f is not a homeomorphism of D onto itself, then the point z_o of lemma 3.8 is a common fixed point for all continuous g on the closure of D which map D into itself and commute with f. (Note: g need not be holomorphic.)

Proof. Let z be in D. Then

$$\begin{aligned} g(z_o) &= g(\lim_{i\to\infty} f^{n_i}(z)) && \text{by lemma 3.8} \\ &= \lim_{i\to\infty} gf^{n_i}(z) && \text{by continuity of } g \\ &= \lim_{i\to\infty} f^{n_i}g(z) && \text{by commutativity} \\ &= z_o && \text{by lemma 3.8.} \end{aligned}$$

Thus the lemma is proved.

Lemma 3.10 (Shields [25]) Let f be a bilinear map of D onto itself. Then either

(1) $f(z) = z$

or

(2) f has exactly one fixed point in the unit disc

or

(3) f has two fixed points on the boundary of D and the iterates of f converge to one of these points.

Proof. We can write the general form of f as

$$f(z) = \alpha\left(\frac{z-a}{1-\bar{a}z}\right) \quad \text{where} \quad |\alpha| = 1, \quad a < 1 .$$ (See Hille [15] V.2, p. 236.)

Suppose f is not the identity map. Then solving the equation $f(z) = z$ for the fixed points we get the equation $\bar{a}z^2-(1-\alpha)z-\alpha a = 0$ which remains unchanged by replacing z by $\frac{1}{\bar{z}}$. So each fixed point is the

reciprocal of the other. Thus either there is one fixed point z_o inside the unit circle and one $\frac{1}{z_o}$ on the outside, or there is a double fixed point on the boundary, or there are two distinct fixed points on the boundary. Since the first two cases give exactly one fixed point in the unit disc we need only consider the last case.

Let p be a bilinear transformation of D onto the upper half of the complex plane which takes one of the fixed points into zero and the other into ∞. Define $g = pfp^{-1}$. Now 0 and ∞ are the fixed points of g which maps the upper half plane into itself and g is bilinear. Thus $g(z)$ must be $g(z) = az$ with $a > 0$. If $a = 1$, then $f(z) \equiv z$ which is (1). If $a \neq 1$ the iterates of g, $g^n(z) = a^n z$ converge either to zero or infinity so the iterates of f converge to one of the two fixed points of f.

<u>Theorem 3.11</u> (Shields [25]) Let F be a commuting family of continuous functions mapping the closed unit disc into itself and analytic in the open unit disc D. Then there is a common fixed point z_o of all the functions in the family.

<u>Proof</u>. If F contains a constant function, then this is the fixed point. If there is no constant function F, then $f(D)$ is contained in D for every f in F. For by the maximum modulus principle if f attained its maximum say z_o with $|z_o| < 1$ then $f(z) \equiv z_o$.

If there is one f in F which is not a homeomorphism of D onto itself then by lemma 3.9 there is a fixed point common to all functions in F.

[illegible]

[illegible]

[illegible]

[illegible]

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If all the functions in F are homeomorphisms of D onto itself then all the functions are bilinear maps (See Hille, V. 1, P. 236 [15]). Thus by lemma 3.10 those function in F not equal to the identity function have one or two fixed points in the closed unit disc.

If there is an f in F with one fixed point then this fixed point is common to all functions in F. Otherwise all the functions in F have two fixed points.

Now if f and g commute it follows that the n^{th} iterate of f, f^n and g commute. So if f is in F we can add the iterates of f to F and still have a commuting family. However by lemma 3.10 the iterates of f converge to one of the fixed points of f and this is the fixed point common to all the functions in F.

CHAPTER IV

PERIODIC POINTS

Consider a pair of continuous functions mapping the unit interval I into itself, say f and g. Suppose f and g commute and that the set $F = \{x | f(x) = x\}$ is finite. Then there is an integer n and an x in F such that $f(x) = x = g^n(x)$. To see this note that g maps F, the set of fixed points of f, into itself. Thus if F has, say m, elements and x is in F the set $\{g^n(x) | n = 0,1,\ldots,m\}$ must have some element x_o listed more than once. Thus $g^n(x_o) = x_o = f(x_o)$ for some $n \leq m$.

The present chapter is based on a paper by A. J. Schwartz [24] in which he shows that if f is continuously differentiable then the above result is valid even if F is infinite. Toward this end we make the following definition. A subset Y of I is called g-invariant if $g(Y)$ is contained in Y.

Schwartz shows that F contains a "minimal" subset which is closed and g invariant. He shows that such a set must be contained in the closure of the intersection of F with the set $P_g = \{x | g^n(x) = x$ for some integer $n\}$. Thus if a minimal g-invariant subset of F is non-empty then the intersection of F with P_g is not empty. So by the definition of P_g and F there is an x in F such that $f(x) = x = g^n(x)$ for some integer n.

Note that in the above paragraph, "minimal" was not defined.

So to make explicit the idea we make the following definition; a closed, non-empty, g-invariant subset of I will be called g-minimal if it contains no proper subset that is closed, non-empty and g-invariant. Notice that a g-minimal set is non-empty, a necessary property for the proof of Schwartz's result.

We will need the next few lemmas in later chapters as well as this one.

Lemma 4.1 The graphs of f and g intersect.

Proof. The proof is by contradiction. If f and g do not intersect then we may assume that $f(x) > g(x)$ for all x in I. Since the set $S = \{x | g(x) \geq x\}$ contains zero and is closed (closure follows from the continuity of g) it has a largest element c which is evidently a fixed point of g. However $f(c)$ is also a fixed point of g since $gf(c) = fg(c) = f(c)$. Thus $f(c)$ is also in S and we must have $f(c) \leq c = g(c)$. So we contradict the assumption $f(x) > g(x)$. Therefore the graphs of f and g intersect.

Lemma 4.2 The set $F = \{x | f(x) = x\}$ is not empty.

Proof. $f(0) \geq 0$ and $f(1) \leq 1$. If $f(0) = 0$ or $f(1) = 1$ we are done. Suppose that $f(0) > 0$ and $f(1) < 1$ then by the intermediate value theorem there is an x in I such that $f(x) = x$.

The following lemma asserts the existence of g-minimal sets

in I , since I is closed, g-invariant and non-empty.

Lemma 4.3 (Schwartz [24]) Every closed, g-invariant, non-empty subset of I contains a g-minimal set.

Proof. Let X be a non-empty, closed, g-invariant subset of I and let

$$J = \{B \subset X \mid B \text{ is non-empty, closed, and g-invariant}\} \quad .$$

Then J is not empty since X is in J . Order J by containment, i.e. for A and B in J , $A > B$ if and only if $A \subset B$. Let β be a chain in J then β has an upper bound namely $C = \bigcap_{B \in \beta} B$. To see this note that,

(a) given any B_o in β , $B_o \supset \bigcap_{B \in \beta} B$ so $C > B_o$

(b) C is closed since it is the intersection of closed sets. It follows from the finite intersection property that C is not empty.

(c) C is g-invariant: for if x is in C then x is in B for every B in β so g(x) is in B for every B in β . Hence g(x) is in C .

So J has a maximal element which is thus a g-minimal set.

Notation: We will denote by $O_g(x)$ the set $\{g^k(x) \mid k \geq 0\}$, called the g-orbit of x (here $g^o(x) \equiv x$) , and by $C_g(x)$ the closure of $O_g(x)$.

Lemma 4.4 (Schwartz [24]) If Y is g-minimal and y is in Y, then $C_g(y) = Y$.

Proof. Clearly $C_g(y)$ is closed and non-empty. Note that $g(O_g(y)) \subset O_g(y) \subset C_g(y)$. Thus by the continuity of g, $g(C_g(y)) \subset C_g(y)$.

Now, since Y is g-minimal, $C_g(y)$ contains Y. However y in Y implies that $O_g(y)$ is contained in Y. So $C_g(y) = \overline{O}_g(y) \subset Y$ since Y is closed. Therefore $C_g(y) = Y$ and the lemma is proved.

Definition. We shall say that x is g-periodic if x is in P_g. A point x in I will be called g-recurrent if x is in $C_g(g(x))$. Thus we see that x is g-recurrent if the set $g^k(x)$ comes arbitrarily close to x.

Lemma 4.5 If Y is g-minimal and $y \in Y$, then y is g-recurrent.

Proof. A rewording of lemma 4.4.

Lemma 4.6 (Schwartz [24]) If Y is g-minimal and it is not the g-orbit of a periodic set then it is a perfect set.

Proof. Since each y in Y is in $C_g(g(y))$ (by lemma 4.5) and not in $O_g(g(y))$, no y is isolated.

Lemma 4.7 (Schwartz [24]) F contains g-minimal sets.

Proof. Let $f_1(x) = f(x)-x$ then $F = f_1^{-1}(0)$ is the inverse image of the closed set $\{0\}$ under the continuous map f_1. Thus F is closed. We have already seen that F is g-invariant. Thus by lemma 4.3 F contains g-minimal sets.

Lemma 4.8 If F is countable then there is an x in I such that $x = f(x) = g^n(x)$ for some integer n.

Proof. Let F be countable. If F contains a g-periodic point, say x_o of period n, then we are done, for $x_o = f(x_o) = g^n(x_o)$. If not then by lemma 4.7 F contains a g-minimal set F' and so by lemma 4.6 F' is a perfect set and hence uncountable, which is a contradiction, hence the result is true.

Theorem 4.9 (Schwartz [24]) Every g-minimal set is contained in the closure of P_g.

Proof. Let Y be a g-minimal set. If Y is the g-orbit of a periodic point then we are done. Suppose then that Y is not a periodic g-orbit. Let $b = \inf Y$. Then since Y is closed b is in Y and by lemma 4.5, $Y = C_g(g(b))$. Thus for a give $\epsilon > 0$, there exist integers N and M such that

$$b < g^{N+M}(b) < g^N(b) < b + \epsilon \quad .$$

Since Y is g-minimal and not a periodic orbit $g^M(b) > b$. By continuity of g^M we have that there is a point in $(b,g^N(b))$ such

that $g^M(e) = e$. ($g^M(b)$ lies above the line $y = x$ and $g^M g^N(b)$ lies below the line $y = x$) . So there are g-periodic points in every neighbourhood of b .

From lemma 4.4 we have that for each y in Y , $|g^K(b)-y| < \frac{\epsilon}{2}$ for some positive integer K . Now by the continuity of g^K there exists a $\delta > 0$ for which $|g^K(x) - g^K(b)| < \frac{\epsilon}{2}$ whenever $|x-b| < \delta$. But we have shown above that there is a point z and an integer L such that $|b-z| < \delta$ and $g^L(z) = z$. Thus $g^K(z)$ is a g-periodic point less than a distance ϵ from y .

Corollary. A minimal set is nowhere dense.

Proof. By theorem 4.9, either Y is contained in P_g or Y is contained in $K = \{x | x$ in the closure of P_g and x not in $P_g\}$. If Y is contained in K , and there is an open G contained in Y , then $G \cap P_g$ is not empty contrary to hypothesis. If Y is contained in P then Y is finite and so nowhere dense.

Up to this point we have not used any assumptions other than continuity and commutativity of f and g in the proofs. Thus the first nine results are valid under the original assumptions. In order to prove lemma 4.10 we need to know that f is differentiable. Furthermore the assumption that f is continuously differentiable is crucial to the proof of theorem 4.11 and, therefore, to the proof that $P_g \cap F$ is not empty.

Lemma 4.10 (Schwartz [24]) If Y is a perfect subset of F, then $f'(y) = 1$ for all y in Y.

Proof. Let $\{y_n\}$ be a sequence in $Y \sim \{y\}$ such that $\{y_n\}$ converges to y. Then

$$f'(y) = \lim_{n \to \infty} \frac{f(y_n)-f(y)}{y_n-y} = \lim_{n \to \infty} \frac{y_n-y}{y_n-y} = 1 .$$

Theorem 4.11 (Schwartz [24]) Let Y be a minimal subset of F. Then Y is contained in the closure of $P_g \cap F$.

Proof. If Y is a periodic orbit then we are done. Suppose Y is not a periodic orbit then by lemma 4.6 it is perfect and so by lemma 4.10 $f'(y) = 1$ for all y in Y.

For a given $\epsilon > 0$ choose y_1 and y_2 in Y and x such that

(1) $y_1 < x < y_2 < y_1+\epsilon$

(2) $g^n(x) = x$ for some n.

This is possible by theorem 4.9. If $f(x) = x$ we have an x in $P_g \cap F$ lying within ϵ of y_1 and y_2. If $f(x)$ does not equal x choose z_1 and z_2 in F so that

(1) $y_1 \leq z_1 < x < z_2 \leq y_2$

(2) (z_1,z_2) contains no point of F.

This is possible since y_1 and y_2 are in F. Thus $f(\omega) > \omega$ for

all ω in (z_1,z_2) or $f(\omega) < \omega$ for all ω in (z_1,z_2) . Suppose $f(\omega) > \omega$, then

(1) $$f(z_2)-f(\omega) = z_2-f(\omega) = f'(\omega')(z_2-\omega) \text{ for some } \omega' \text{ in } (\omega,z_2) .$$

Since f' is a continuous function we may assume that the ω given is small enough so that $|f'(u)-f'(v)| < \frac{1}{2}$ whenever $|u-v| < \epsilon$. Thus

$$f'(\omega') > f'(z_2) - \frac{1}{2} = \frac{1}{2} \text{ since } z_2 \text{ is in } Y \text{ which is contained in } F .$$

Hence from (1) we have $f(\omega) < z_2$.

Thus $\{f^k(x)\}$ is an increasing sequence in (z_1,z_2) with limit ℓ . Since $f(\ell) = \lim_{k \to \infty} f^{k+1}(x) = \ell$, ℓ is in F . In fact $\ell = z_2$. But $g^n f^k(x) = f^k g^n(x) = f^k(x)$ for each k so

$$\lim_{k \to \infty} g^n f^k(x) = \lim_k f^k(x)$$

therefore $g^n \lim_{k \to \infty} f^k(x) = \ell$

$$g^n(\ell) = \ell .$$

Therefore we have found an element ℓ such that $g^n(\ell) = \ell$ and $f(\ell) = \ell$ lying within ϵ of y_1 and y_2 . But y_1 was arbitrary so y is in the closure of $P_g \cap F$.

<u>Theorem 4.12</u> (Schwartz [24]) Suppose that g is continuous and that f is continuously differentiable, then there is a point x and an

integer n such that $x = f(x) = g^n(x)$.

<u>Proof</u>. By lemma 4.7, F contains a minimal set Y , which by definition is non-empty. Thus according to theorem 4.11 there is a point y in $P_g \cap F$. Now it follows from the definition of P_g and F that $x = f(x) = g^n(x)$ for some integer n .

CHAPTER V

NON-CYCLIC FUNCTIONS

In certain circumstances it is easy to see that f and g must have a common fixed point. For example suppose that for every closed C of I such that $f(C)$ is contained in C, f has a fixed point in C. Then it follows that f and g have a common fixed point since G, the set of fixed points of g, is closed and $f(G)$ is contained in G. Also it is clear that if for each point x in I the sequence $\{f^n(x)\}$ is convergent, then f and g have a common fixed point.

It is the purpose of this chapter, which is based on a paper by Chu and Moyer [9], to show a number of conditions to be equivalent to those in the above paragraph. Then we will use these relations to show that if there is a subinterval C of I such that $f(C)$ is contained in C, $f(x) \neq x$ implies $f^2(x) \neq x$ for every x in C, and g has a fixed point in C, then f and g have a common fixed point in C and hence in I. Working independently of the above authors Maxfield and Mourant [21] also obtained this result.

To this end consider the following

<u>Theorem 5.1</u> (Chu and Moyer [9]) Let f be a continuous map of I into itself. Let $F_n = \{x | x = f^n(x)\}$, $F_n^+ = \{x | f^n(x) > x\}$ and $F_n^- = \{x | f^n(x) < x\}$. Then the following conditions are equivalent.

(i) (a) $F_1 = F_2$

(b) $F_1^+ = F_2^+$ and $F_1^- = F_2^-$

(ii) If C is any non-empty closed subset of I mapped into itself by f then f has a fixed point in C.

(iii) (a) $F_1 = F_i$ for all i

(b) $F_1^+ = F_i^+$ and $F_1^- = F_i^-$ for all i

(iv) $O_f(x)$ is a convergent sequence for every x in I.

Proof. We first show that (iii) (a) is equivalent to (iii) (b) and thus (i) (a) is equivalent to (i) (b).

That (iii) (b) implies (iii) (a) is immediate.

To show (iii) (a) implies (iii) (b) : If x is in F_1^+, then by (iii) (a), x is not in F_k for any k. For the sake of an argument by contradiction suppose that there is a positive integer k such that x is in F_k^-.

There are two cases to consider: either $[0,x]$ is contained in F_1^+ or $F_1 \cap [0,x]$ is not empty. If $[0,x]$ is contained in F_1^+ then 0 is contained in F_k^+, $k = 2,3,\ldots$. For $f^k(0) \geq 0$ and by (iii) (a) $f^k(0) \neq 0$ for $k = 2,3,\ldots$. Now f^k is continuous so there must be a point x_o common to both $[0,x]$ and F_k. But this contradicts the assumption that $[0,x]$ is contained in F_1^+ since by (iii) (a) $f(x_o) = x_o$.

If $F_1 \cap [0,x]$ is not empty, then it has a largest element, say x_o Since $f(x) > x$, (x_o,x) is contained in F_1^+. Now f is

continuous so there is an $h_1 > 0$ such that f maps (x_o, x_o+h_1) into (x_o, x) . Thus (x_o, x_o+h_1) is contained in F_2^+ .

We wish to show that for each i in N there is an h_i such that (x_o, x_o+h_i) is contained in F_{i+1}^+ . For if this were the case, F_k and (x_o, x) would have a point in common since $f^k(y) > y$ whenever y is in (x_o, x_o+h_{k+1}) and $f^k(x) < x$. But this would contradict (iii) (a) and the assumption that x_o is the largest fixed point for f in $[0,x]$.

We have already shown that h_1 exists. Let us suppose, for an induction argument that h_k exists. Then there is an h_{k+1} such that f maps (x_o, x_o+h_k) into (x_o, x_o+h_{k+1}) . Thus

$$f^{k+2}(y) = f(f^{k+1}(y)) > f(y) > y \quad ,$$

for each y in (x_o, x_o+h_{k+1}) . Therefore h_i exists for each i in N . Hence we arrive at the contradiction mentioned above. So x is in F_k^+ . A similar argument will show that if x is in F_1^- then x is in F_k^- . Thus (iii) (a) is equivalent to (iii) (b) .

To show that (i) implies (ii) we will suppose (ii) is false and show that (i) must be false. If (ii) is false then there is a non-empty closed subset C of I such that $f(C)$ is contained in C and for which the intersection with F_1 is empty. Since $C \cap F^+$ is a closed set, it has a largest element. Call this element x_o .

We first show that f maps $[x_o, f(x_o)]$ into $(x_o, b]$ so that the image of $[x_o, f(x_o)]$ under f is a compact subset of $(x_o, b]$.

Let x_1 be the smallest fixed point of f that is greater than x_o. This point exists since $f(x_o) > x_o$ and $f(b) \leq b$. Then (x_o,x_1) is contained in F^+. Thus if $f(x_o) \leq x_1$, then $f([x_o,f(x_o)])$ is contained in $(x_o,b]$. So we assume that $f(x_o) > x_1$. Since f is continuous we must have that $[x_1,f(x_o)]$ is contained in $f([x_o,x_1])$. Therefore $f([x_1,f(x_o)])$ is contained in $f^2([x_o,x_1])$. But by (i) (b), $f^2([x_o,x_1]) \subset (x_o,b]$. So we have that $f([x_1,f(x_o)])$ is contained in $(x_o,b]$ and therefore f maps $[x_o,f(x_o)]$ into $(x_o,b]$.

We next show by induction that

$$f(x_o) \geq f^2(x_o) \geq \dots f^k(x_o) \geq f^{k+1}(x_o) \geq x_o \qquad (1)$$

for all positive integers k. Note that $f(C)$ is contained in C so that $f^k(x)$ is an element of C for each integer k. Now $f^2(x_o) > x_o$ by (i) (b) and $f^2(x_o) = f(f(x_o)) \leq f(x_o)$ by choice of x_o so (1) holds for $k = 1$. Suppose (1) holds for $k = N$, then we have

$$x_o < f^{N+1}(x_o) \leq f^n(x_o) \leq f(x_o) \quad .$$

However x_o was chosen to be the largest element of C for which $f(x) > x$. Therefore $f^{N+2}(x_o) \leq f^{N+1}(x_o)$. But, $f^{N+1}(x_o)$ is an element of $[x_o,f(x_o)]$. Therefore, since f maps $[x_o,f(x_o)]$ into $(x_o,b]$ we have

$$x_o < f^{N+2}(x_o) \quad .$$

Hence by induction (1) holds for all k. This means that the sequence $\{f^k(x)\}$ is monotone and therefore convergent to some point x_2. Since

C is closed, x_2 is in C. Furthermore by the continuity of f, $f(x_2) = x_2$ contrary to the assumption that f has no fixed point in C.

To show that (ii) implies (iii) we will use proof by contradiction. Assume that there is an x in I and a $k > 1$ such that $f(x) \neq x$ and $f^k(x) = x$. Then the set $O_f(x)$ is finite hence, closed. Furthermore it is invariant under f so by (ii) it has a fixed point which by assumption is not x. Hence there must exist an m, $1 < m < k$, such that $f(f^m(x)) = f^m(x)$. Therefore

$$x = f^{k-m}(f^m(x)) = f^{k-m}(f(f^m(x)) = f(x)$$

contrary to assumption. Thus (ii) implies (iii).

To show that (iii) implies (iv) let x in I be given. Then $f(x) > x$, or $f(x) = x$ or $f(x) < x$. If $f(x) = x$ then $f^k(x) = x$ for each k in N and we are done.

Suppose $f(x) > x$. Since f maps I into I, $\{f^k(x)\}$ has at least one accumulation point. First we show that the sequence cannot have more than two accumulation points.

Suppose a, b and c are accumulation points for the sequence $\{f^k(x)\}$ with $a < b < c$.

Case 1: $f(b) > b$ (or $f(b) < b$). Since $f(b) > b$ and f is continuous there is a neighbourhood $N(b)$ of b contained in $(b - \frac{f(b)-b}{2}$, $b + \frac{f(b)-b}{2})$ such that $f(y) > \frac{f(b)+b}{2} > b$ for every y in $N(b)$.

Now b is an accumulation point of the sequence $\{f^k(x)\}$ so there is an integer m such that $f^m(x)$ is in $N(b)$. Thus $f(f^m(x)) > f^m(x)$. Therefore $f^{k+m}(x) = f^k(f^m(x)) > f^m(x)$ (by (iii)) for all $k = 2,3,\ldots$. Hence a is not an accumulation point for $\{f^k(x)\}$. Similarly c is not an accumulation point for $\{f^k(x)\}$ if $f(b) < b$.

Case 2: $f(b) = b$. It is clear that $f(f^k(x)) < f^k(x)$ whenever $f^k(x) > b$ and $f(f^k(x)) > f^k(x)$ whenever $f^k(x) < b$. If this were not the case, say for example, that for some $k > 1$ $f^k(x) < b$ and $f(f^k(x)) < f^k(x)$. Then by (iii) $f^{m+k}(x) < f^k(x)$ for all m. So b would not be an accumulation point.

Since b is an accumulation point there is an integer k such that $f^k(x)$ is in $[b-\delta,b+\delta]$ where $\delta = \min\{\frac{c-b}{2}, \frac{b-a}{2}\}$. If $f^k(x) \neq b$, say $f^k(x) > b$, then $f(f^k(x)) < f^k(x)$ so

$$f^{m+k}(x) < f^k(x) < \frac{b+c}{2} \quad \text{for all } m .$$

Thus c is not an accumulation point of $O_f(x)$. Similarly $f^k(x) < b$ implies that a is not an accumulation point of $O_f(x)$. Therefore $f^k(x) = b$ for each k such that $f^k(x)$ is in $[b-\delta,b+\delta]$. Since b is an accumulation point there are at least two integers k_1, k_2 with $k_1 < k_2$ such that $f^{k_1}(x)$, $f^{k_2}(x)$ are in $[b-\delta,b+\delta]$. Hence $f^{k_2-k_1}(f^{k_1}(x)) = f^{k_1}(x)$, so by (iii) (a) $f^{m+k_1}(x) = b$ for all m. Therefore a and c are not accumulation points of $O_f(x)$. Thus there cannot be more than two accumulation points for $O_f(x)$.

Suppose there are exactly two accumulation points of $O_f(x)$. Namely a and b with $a < b$. Then as we have seen above, we cannot have $a < f^k(x) < b$ for any k. Furthermore, whenever $f^k(x) < a$ $f(f^k(x)) > f^k(x)$ and whenever $f^k(x) > b$, $f(f^k(x)) < f^k(x)$. Let $\{f^{k_i}(x)\}$ be the subsequence of $O_f(x)$ contained in $[0,a]$ with $k_i < k_{i+1}$ for all i.

Since b is an accumulation point of $O_f(x)$ there is a subsequence of $\{f^{k_i}(x)\}$, say $\{f^j(x)\}$ such that $f(\{f^j(x)\})$ is contained in $[b,1]$. Otherwise there are only a finite number $f^{k_i}(x)$ such that $f(f^{k_i}(x))$ is an element of $[b,1]$. So there is a k_i for which $f^m(f^{k_i}(x))$ is contained in $[b,1]$ for all positive integers m. But this means that there is only a finite number of elements in $[0,a]$ contrary to the assumption that a is an accumulation point of $O(x)$.

Now $\{f^j(x)\}$ must converge to a, and $\{f(f^j(x)\}$ must converge to b. f is continuous so $f(a) = b$. Similarly $f(b) = a$. Thus $f^2(a) = a$ which by (iii) is only possible if $a = b$. Hence (iii) implies (iv).

To see that (iv) implies (i) note that if there is an x such that $f(x) \neq x$ but $f^2(x) = x$, the sequence $\{f^k(x)\}$ cannot converge. Thus our theorem is proved.

<u>Definition</u>. Let f be a continuous function mapping an interval $[a,b]$ into itself. If $f(x) \neq x$ implies $f^2(x) \neq x$ for all x in

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[a,b], then f is called <u>non cyclic</u>.

The following theorem shows that the commuting functions conjecture holds true for another special case.

<u>Theorem 5.2</u> (Chu and Moyer [9]) Let f and g be two continuous functions of I into itself such that f and g commute on I. Suppose there is a subinterval $[a,b]$ of I on which one of the functions say f, is non cyclic, and in which g has a fixed point. Then f and g possess a common fixed point in $[a,b]$ and thus in $[0,1]$.

<u>Proof</u>. Let G be the set of fixed points of g in $[a,b]$. Then G is non-empty by hypothesis, $f(G)$ is contained in G since f and g commute and G is closed since g is continuous. Hence (i) (a) is satisfied so by Theorem 5.1, f has a fixed point in G. Thus f and g have a common fixed point.

CHAPTER VI

LIPSCHITZ FUNCTIONS

The purpose of this chapter is to verify the commuting function conjecture when one or both of the pairs of commuting functions satisfy certain Lipschitz conditions. The work of De Marr [11] and Jungck [20] forms a basis for the material in this chapter.

In the remainder of this chapter we will assume that f and g are continuous functions on I which commute under composition. The following definition will make the exposition of De Marr's work somewhat simpler. The function f is called α-Lipschitz on I if for every x and y in I, $|f(x)-f(y)| \leq \alpha\ |x-y|$. It is clear that if $\alpha < 1$, then f must have exactly one fixed point. Thus if f and g commute the fixed point of f must be the fixed point of g; for g maps F, the set of fixed points of f, into itself. The following lemma proves the conjecture when $\alpha = 1$.

Lemma 6.1. If f is α-Lipschitz with $\alpha = 1$, then f and g have a common fixed point.

Proof. If f has two fixed points say x and y with $x < y$ then the interval $[x,y]$ is contained in the set of fixed points of f. To see this suppose there is an element z in (x,y) which is not a fixed point of f, then $f(z) > z$ or $f(z) < z$. If $f(z) > z$,

then $|f(x)-f(z)| > |x-z|$ contrary to the assumption that $\alpha = 1$. Similarly we reach a contradiction if we assume $f(z) < z$.

Therefore F must be an interval or a single point. The single point as we have seen leads to a common fixed point for f and g. In the case where f is an interval we have that $g(F) \subset F$ so that g must have a fixed point in F by the intermediate value theorem.

Thus the conjecture is verified for $0 \leq \alpha \leq 1$. One might wonder then, how large can we make α before having to place additional restrictions on g (if any). In other words, what restrictions on g will be sufficient when α is large. One answer was given by De Marr [11] in the following theorem.

<u>Theorem 6.2</u> (De Marr [11]) Suppose that f and g are α-Lipschitz and β-Lipschitz respectively. If $\alpha > 1$ and $\beta < \frac{\alpha+1}{\alpha-1}$ then f and g have a common fixed point.

<u>Proof</u>. The proof is by contradiction. Let G be the set of fixed points of g. Then as we have seen before G is closed and $f(G) \subset G$. Thus there is in G a smallest element a and a largest element b. Assuming that f and g have no common fixed points we have that $f(a) > a$ and $f(b) < b$. Thus the set $A = \{(x,y) | x,y \in G,\ x < f(x), y > f(y)\}$ of ordered pairs is not empty and it is closed in the product topology on $I \times I$. So if $c = \inf\{|y-x| \,|\, (x,y) \in A\}$ there is at least one pair (x_o,x_1) in A such that $x_1 - x_o = C$. This pair determines an interval $I_o = [x_o,x_1]$.

Evidently there are no elements of G between x_o and x_1. For if z is a fixed point of g, $f(z) \neq z$ by assumption so $f(z) < z$ or $f(z) > z$. Consider the case where $f(z) < z$. Then z is in G, $f(z) < z$ and $x_o < z$, but $z - x_o < C$ contrary to the definition of C. Similarly $f(z) > z$ leads to a contradiction. Thus if we let $y_o = f(x_o)$ and $y_1 = f(x_1)$ we have

$$y_o \geq x_1 \quad \text{and} \quad y_1 \leq x_o \tag{1}$$

Since x_o and x_1 are the only elements of G in I and since g is continuous it follows that $g(x) > x$ for all x in the interior of I_o or $g(x) < x$ for all x in the interior of I_o. Let us consider the case where $g(x) < x$.

Since $x_o < f(x_o)$ and $x_1 > f(x_1)$ we must have a fixed point in I_o. Let s' be the smallest fixed point of f in I_o and let $t' = g(s')$. Now t' is a fixed point of f and $s' > g'(s) = t'$ and $x_o > t'$. So we have from these inequalities, (1) and the Lipschitz conditions, the following inequalities

(a) $\quad x_1 - t' \leq f(x_o) - t' \qquad$ since $f(x_o) > x_1$

$\qquad = f(x_o) - f(t') \qquad f(t') = t'$

$\qquad \leq \alpha(x_o - t') \qquad f$ is α-Lipschitz

(b) $\quad x_1 - t' = g(x_1) - t'$

$\qquad = g(x_1) - g(s')$

$\qquad \leq \beta(x_1 - s')$

(c) $$x_1 - s' = x_1 - f(s') \le f(x_o) - f(s') \le \alpha(s' - x_o) \quad .$$

From (a) we get

$$\alpha x_o - x_1 \ge t'(\alpha - 1)$$

$$\ge (1-\alpha)(\beta s' + x_1(1-\beta)) \qquad \text{substituting for } t' \text{ from (b)} .$$

Which after a little algebra becomes

(d) $$\alpha(x_1 - x_o) \le \beta(\alpha-1)(x_1 - s') \quad .$$

Now

$$x_1 - s' \le \alpha(s' - x_o) \qquad \text{(c)}$$

so

$$x_1 - s' \le \alpha[(s'-x_1) + (x_1-x_o)]$$

which can be rewritten as

$$(1+\alpha)(x_1-s') \le \alpha(x_1-x_o)$$

$$\le \beta(1-\alpha)(s'-x_1) \qquad \text{by d}$$

$$= \beta(\alpha-1)(x_1-s') \quad .$$

Hence

$$\frac{1+\alpha}{\alpha - 1} \le \beta \quad .$$

However this contradicts our hypothesis. Thus f and g must have a common fixed point.

Suppose that $|f(x)-f(y)| \leq |x-y| + k(x,y)$ for every x and y in I with k being a non-negative real valued function on $I \times I$. We showed, at the beginning of this chapter that if f was (1) Lipschitz then f and g must have a common fixed point. Thus if $k(x,y) \equiv 0$, f and g have a common fixed point. The question arises as to whether there are any other functions $k(x,y)$ which guarantee a common fixed point. The second result in this chapter, due to Jungck, gives one class of such functions. He shows that the family consisting of $k(x,y)$ of the form $k(x,y) = \alpha|gf(x) - gf(y)|$ with $\alpha > 0$ is a class which guarantees a common fixed point. To prove this we will need the following:

<u>Lemma 6.3</u> (Jungck [20]) If f and g have no common fixed point then there exist a and b in I such that

(a) $f(a) = g(a) \geq b > a \geq f(b) = g(b)$

(b) $g(x) \neq f(x)$ for all x in (a,b).

<u>Proof</u>. Let $A = \{x \text{ in } I \mid f(x) = g(x)\}$. Then as we have seen A is not empty. If x is in A then $ff(x) = fg(x) = gf(x)$ so $f(x)$ is in A. Similarly $g(x)$ is in A. Since A is closed it has a minimum and a maximum element which we denote by c and d respectively. Since $f(c)$ is also in A $f(c) \geq c$, and $g(c) = f(c) \geq c$. But f

and g have no common fixed point so we must have

$$(1) \qquad f(c) = g(c) > c$$

and similarly

$$(2) \qquad f(d) = g(d) < d \quad .$$

Thus the set $S = \{x \text{ in } I \mid f(x) = g(x) \geq x\}$ is not empty for it contains c . S is closed since f and g are continuous. Thus S has a maximum element "a" . Note $a < d$ otherwise "a" would be a common fixed point of f and g . Thus

$$(3) \qquad f(a) = g(a) > a \ .$$

Similar reasoning yields a minimum element b in $[a,b]$ for which

$$(4) \qquad f(b) = g(b) < b \quad .$$

Clearly $a < b$.

We see that by our choice of a and b that $A \cap (a,b)$ is empty. Thus we have proved part (b).

Since b is in A so is $f(b)$ and by (4) $f(b) < b$ but A has no points in common with (a,b) . Thus $f(b) \leq a$. Similarly we can show that $f(a) \geq b$. So we have $f(a) \geq b > a \geq f(b)$, which from (3) and (4) becomes $g(a) = f(a) \geq b > a \geq f(b) = g(b)$ which was to be proved.

<u>Theorem 6.4</u> (Jungck [20]) f and g have a common fixed point when-

ever there is a real number $\alpha > 0$ such that

$$|f(x)-f(y)| \leq \alpha|gf(x)-gf(y)| + |x-y| \quad \text{for all } x,y \text{ in } I .$$

Proof. Suppose f and g have no common fixed point. Then by Lemma 6.3 there are points a,b in I such that

(a) $\quad f(a) = g(a) \geq b > a \geq f(b) = g(b)$

(b) $\quad f(x) \nmid g(x)$ for all x in (a,b) .

Since f and g are continuous we may assume without loss of generality that $f(x) < g(x)$ in (a,b) . Let d be the smallest fixed point of g in (a,b) . Since f maps the set of fixed points into itself $f(d)$ is a fixed point for g . Now $f(d) < g(d) = d$, and d is the smallest fixed point of g in (a,b) so $f(d) < a < f(a)$. Thus there is an x in (a,d) such that $f(x) = a$ and hence a maximum element c in (a,d) such that $f(c) = a$. Clearly

(1) $\quad f(x) < a$ for all x in $[c,d]$.

Furthermore $f(c) = a$ implies that $gf(c) = g(a) = f(a)$, and since

(2) $\quad f(a) \geq b > d$

we have that $gf(c) > d$. $f(d)$ is a fixed point of g and $f(d) < d$ so $gf(d) < d$. Hence $gf(d) < d < gf(c)$. Thus the continuity of gf guarantees the existence of a point y in (c,d) for which $gf(y) = d$. Moreover the inequality $f(c) = a < d < f(a)$ (from (2)) yields a point x in (a,c) such that $f(x) = d$. Hence

(3) $\quad gf(x) = g(d) = d$.

Notice that y in (c,d) implies by (1) that $f(y) < a$, and since $f(x) = d$ we can write $f(x)-f(y) > d-a$. But $a < x < y < d$ implies that $d-a > y - x > 0$ so that $|f(x)-f(y)| > |x-y|$. Now $gf(x) = gf(y)$ so $|f(x)-f(y)| > \alpha|gf(x)-gf(y)| + |x-y|$ for any positive real α contrary to our assumption. Thus the theorem is proved.

CHAPTER VII

PROPERTIES OF THE FIXED POINTS OF THE COMPOSITE FUNCTION

In the proofs of some of the previous theorems we found it necessary to know something about the sets $f(H)$, $g(H)$, $f(G)$ and $g(F)$, where F, G, H are the sets of fixed points of f, g and $fg = h$, respectively. The purpose of this chapter is to give further consideration to these sets, with $f(H)$ and $g(H)$ being of particular interest.

It is Baxter's [2] work and Baxter and Joichi's [3] work which is a basis for the material in the first and second parts of this chapter respectively. First following Baxter's idea we will develop some of the properties of the functions $f|H$ and $g|H$. These turn out to be permutations of H. Then in the second part of this chapter we consider properties of permutations on H having certain of the properties possessed by $f|H$ and $g|H$.

Before doing this, however, let us point out some of the characteristics of $f(G)$, $f(H)$, $g(F)$, and $g(H)$. We have seen that $f(G)$ is contained in G and $g(F)$ is contained in F. Furthermore it follows from the definition of H that

Property 1: $f|H$ and $g|H$ are inverse permutations on H. i.e. $g(f(x)) = f(g(x)) = x$ for all x in H.

Property 2: $h|F = g|F$ and $h|G = f|G$.

Proof. Let x be in F then $h(x) = gf(x) = g(x)$. A similar argument holds for $h|G$.

Note that Property 2 implies that h maps F and G into F and G respectively.

Property 3: If F and H have a point in common then so do F and G and so f and g have a common fixed point.

Proof. Let x be common to both F and H . Then

$$x = h(x) = gf(x) = g(x) \text{ so } x \text{ is in } G .$$

Part A

Following Baxter's idea we will assume throughout this chapter that the set of fixed points of the composite function H is finite. Let $H = \{x_1, x_2, \ldots, x_n\}$ with $x_i < x_{i+1}$. Then, as Baxter did, we can classify the intervals $I_i = [x_i, x_{i+1}]$ as either up intervals or down intervals depending on whether $h(x) > x$ or $h(x) < x$ for all x in I_i . We can also classify the points x_i for $2 \leq i \leq n-1$ as being

(a) up-crossing: if I_{i-1} and I_i are down- and up-intervals respectively

(b) down-crossing: if I_{i-1} and I_i are up- and down-intervals respectively

(c) touching: if I_{i-1} and I_i are of the same type.

Furthermore we will call

(d) the point x_1 down-crossing or touching according as I_1 is a down or up interval respectively

(e) the point x_n down-crossing or touching according as I_{n-1} is an up-interval or down interval respectively.

We remark at this point that if f and g are differentiable then $h'(f(x_o)) = f'(x_o)g'f(x_o) = h'(x_o)$ whenever x_o is in H. This suggests that f and g permute the fixed points of each type. Our aim is for a more general result. We want to show that in fact f and g permute the fixed points of each type, and that the assumption, f and g are differentiable on I is not necessary. To do this we will need the following lemmas.

<u>Lemma 7.1</u> (Baxter [2]) Let $f(x_m)$ belong to $f(I_k)$ for some x_m. Then $x_m \geq x_k$ if I_k is an up-interval and $x_m \leq x_{k+1}$ if I_k is a down-interval.

<u>Proof.</u> Consider the case where I_k is an up-interval and let "a" be an element of I_k such that $f(x_m) = f(a)$. Then since x_m is in H we have $x_m = gf(x_m) = gf(a) \geq a \geq x_k$. A similar argument will give the result for the case when I_k is a down-interval.

<u>Lemma 7.2</u> (Baxter [2]) If $f(x_k)$ and $f(x_{k+1})$ are successive fixed points, say x_m and x_{m+1} and if I_k and I_m are of different

types then $f(I_k) = I_m$ and $g(I_m) = I_k$.

<u>Proof</u>. Assume that I_k is an up-interval and suppose that $f(I_k) \nsubseteq I_m$. Then by the intermediate value theorem there is an a_o in the open interval (x_k, x_{k+1}) such that $f(a_o) = x_m$ or $f(a_o) = x_{m+1}$. If $f(a_o) = x_m$ then $a_o < h(a_o) = gf(a_o) = g(x_m) = x_k$. But this contradicts the assumption that a_o is in (x_k, x_{k+1}) . Thus $f(x) > x_m$ for all x in (x_k, x_{k+1}) .

If $f(a_o) = x_{m+1}$, then since a_o is in the interval $(g(m), g(m+1))$ we have by the intermediate value theorem that there is a b_o in the interval (x_m, x_{m+1}) such that $g(b_o) = a$. So $b_o > fg(b_o) = f(a_o) = x_m$ since I_m is a down interval. But this contradicts the assumption that b_o is in (x_m, x_{m+1}) . So $f(I_k) = I_m$ and $g(I_m) = I_k$.

If I_k is a down-interval the proof is similar. Thus $f(I_k) = I_m$ and $g(I_m) = I_k$ in any case.

<u>Lemma 7.3</u> (Baxter [2]) If $f(x_k)$ and $f(x_{k+1})$ are successive fixed points, say x_m and x_{m+1} , in some order, then I_k and I_m are of the same type when $f(x_k) = x_m$ and I_k and I_m are of different types if $f(x_k) = x_{m+1}$.

<u>Proof</u>. Suppose $f(x_k) = x_m$ and assume contrary to the above, that I_k and I_m are not alike. Since the permutation of H by g is the inverse of the permutation by f , we may assume that I_k is an

up-interval and I_m is a down interval for if this is not the case rename f and g calling g,f and f,g . Let a_o be an arbitrary element in the open interval (x_k, x_{k+1}) . Define

$$(1) \qquad b_o = f(a_o),\ a_{n+1} = g(b_n),\ b_{n+1} = f(a_{n+1}) \qquad n = 0,1,2,\ldots \ .$$

By lemma 7.2 we have that a_n is in I_k and b_n is in I_m for all n . Furthermore, since I_k is an up interval $a_{n+1} = gf(a_n) = h(a_n) > a_n$ while $b_{n+1} = fg(b_n) < b_n$ since I_m is a down interval. Thus $\{a_n\}$ is an increasing sequence of points in I_k approaching x_{k+1} and $\{b_k\}$ is a decreasing sequence of points in I_m approaching x_m . Thus by (1) we have $x_{k+1} > \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} g(b_n) = g(x_m) = x_k$ which is impossible. Hence I_k and I_m are alike. The case where $f(x_k) = x_{m+1}$ is similar.

<u>Lemma 7.4</u> (Baxter [2]) If x_k is an up-crossing point, then $f(x_k) = x_m$ must lie between $f(x_{k-1})$ and $f(x_{k+1})$.

<u>Proof</u>. Note that k does not equal 1 or n since neither x_1 nor x_n can be up-crossing points. Thus x_{k-1} and x_{k+1} exist whenever x_k is an up-crossing.

Since x_k is an up-crossing point, I_{k-1} is a down interval and I_k is an up-interval so $h(I_{k-1}) \subset [0,x_k]$ and $h(I_k) \subset [x_k,1]$. Thus $gf(I_{k-1})$ and $gf(I_k)$ have at most a single point in common . So, $f(I_{k-1})$ and $f(I_k)$ have at most a single point in common. Therefore $f(x_k)$ must lie between $f(x_{k-1})$ and $f(x_{k+1})$ otherwise

$f(I_{k-1})$ and $f(I_k)$ would have an interval in common.

It is a direct consequence of the last lemma that $f(x_k)$ does not equal x_1 or x_n whenever x_k is an up-crossing point and in particular if x_k is an up-crossing point $f(x_k)$ does not equal either 0 or 1 since 0 and 1 would necessarily be x_1 and x_n respectively.

Lemma 7.5 (Baxter [2]) If x_k is a down crossing and if $f(x_k) = x_m$ does not lie between $f(x_{k-1})$ and $f(x_{k+1})$, then either x_{m-1} or x_{m+1} occurs as a value of $f(x_{k-1})$ or $f(x_{k+1})$.

Proof. Consider the case where $f(x_k)$ is less than each of $f(x_{k-1})$ and $f(x_{k+1})$. Then x_{m+1} is an element of both $f(I_{k-1})$ and $f(I_k)$. Let x_j be an element of H such that $f(x_j) = x_{m+1}$. Then $f(x_j)$ is in $f(I_{k-1})$ and $f(x_j)$ is in $f(I_k)$. By lemma 7.1 $x_j \geq x_{k-1}$ and $x_j \leq x_{k+1}$. Since $x_j \neq x_k$ we have that $x_j = x_{k+1}$ or $x_j = x_{k-1}$. The case where $f(x_k)$ is greater than each of $f(x_{k-1})$ and $f(x_{k+1})$ is similar.

Theorem 7.6 (Baxter [2]) Let h have a finite number of fixed points. Then f and g permute the fixed points of each type.

Proof. Since the permutation resulting from f is the inverse of the permutation from g, we need consider only half the possible cases. For example the case (1) where x_k is an up-crossing point and $f(x_k)$ is a down crossing point, is equivalent to the case where

x_k is a down crossing point and $f(x_k)$ is an up crossing point. Two other cases we must consider are case (2) x_k is an up-crossing point and $f(x_k)$ is a touching point, and case (3) x_k is a down crossing point and $f(x_k)$ is a touching point.

Case (1) x_k is an up-crossing point and $f(x_k)$ is a down crossing point. As noted in the proof of lemma 7.4, $1 < k < n$. We shall show that $f(x_{k-1}) < x_m$ and $f(x_{k+1}) < x_m$ which contradicts lemma 7.4. By the remark following lemma 7.4 we have that $x_m \neq x_1, x_n$. Furthermore I_{k-1} and I_m are down intervals while I_k and I_{m-1} are up intervals. Suppose $f(x_{k-1}) > x_m$. Then $f(I_{k-1})$ contains I_m. Since I_{k-1} is a down interval we have $[0,x_k]$ contains $h(I_{k-1}) = gf(I_{k-1})$. But $gf(I_{k-1})$ contains $g(I_m)$ and, since $g(x_{m+1})$ is an element of H we must have $g(x_{m+1})$ in $g(I_m)$. However by lemma 7.5 this implies that $f(x_{k-1}) \leq x_{m+1}$ (for I_m is a down interval). Therefore $f(x_{k-1}) = x_{m+1}$. But then by lemma 7.3 I_{k-1} and I_m are of different types and we have our contradiction.

If $f(x_{k+1}) > x_m$, then $f(I_k) \supset I_m$. Since I_k is an up-interval $[x_k,1] \supset gf(I_k) \supset g(I_m)$ which implies $g(I_m) \supset I_k$. Hence x_{k+1} is in $g(I_m)$ so by lemma 7.1, $f(x_{k+1}) \leq x_{m+1}$. As before we must have $f(x_{k+1}) = x_{m+1}$, which leads to the contradiction I_k and I_m are alike. Thus $f(x_{k+1})$ is not greater than x_m.

From the previous two paragraphs we have, that neither $f(x_{k-1}) > x_m$ nor $f(x_{k+1}) > x_m$, a result which contradicts lemma 7.4. Thus we cannot have the case where x_k is an up-crossing point and $x_k = f(x_k)$ is a down crossing point.

Case 2. x_k is an up-crossing point and $f(x_k)$ is a touching point. By lemma 7.4 we have again that $x_m \neq x_1$ or x_n. In the proof of case 1 we showed that neither $f(x_{k-1}) > x_n$ nor $f(x_{k+1}) > x_m$, if I_m is a down interval. A similar proof will give the result that neither $f(x_{k-1}) < x_m$ nor $f(x_{k+1}) < x$ if I_{m-1} is an up-interval. If x_m is a touching point then I_{m-1} and I_m are alike. Thus either I_m is a down-interval or I_{m-1} is an up-interval. In either case x_m cannot be between $f(x_{k-1})$ and $f(x_{k+1})$ giving a contradiction. Thus we cannot have x_k an up-crossing point and x_m being a touching point.

Case 3. x_k a down crossing point and $x_m = f(x_k)$ an up-crossing point. Suppose first that $f(x_{k-1}) = x_{m+1}$. Then by lemma 7.3 I_m is a down interval. Also, $g(I_{m-1})$ contains either I_{k-1} or I_k so that either $x_{k-1} = h(x_{k-1}) = gf(x_{k-1})$ is in $g(I_{m-1})$ or $x_{k+1} = gf(x_{k+1})$ is in $g(I_{m-1})$. By lemma 7.1, either $f(x_{k-1}) \leq x_m$ or $f(x_{k+1}) \leq x_m$. However $f(x_{k-1}) \leq x_m$ is ruled out by the assumption that $f(x_{k-1}) = x_{m+1}$. If $f(x_{k+1}) < x_m$ then $f(I_k) \supset I_{m-1}$. But this means that $[0,x_{k+1}] \supset h(I_k) \supset g(I_{m-1})$. Thus either $g(x_{m-1}) = x_{k+1}$ or $g(I_{m-1}) \supset I_{k-1}$. Since by lemma 7.3 I_{m-1} and I_k would be of different types if $g(x_{m-1}) = x_{k+1}$, this possibility is eliminated. If we assume that $g(I_{n-1}) \supset I_{k-1}$ then x_{k-1} would be in $g(I_{m-1})$. Thus we arrive at the contradiction $f(x_{k-1}) \leq x_m$ through lemma 7.1. Therefore $f(x_{k-1}) = x_{m+1}$. The cases $f(x_{k-1}) = x_{m-1}$, $f(x_{k+1}) = x_{m-1}$ and $f(x_{k+1}) = x_{m+1}$ can be elminated in a similar manner.

Thus by lemmas 7.4 and 7.5 we have only to consider the case

in which x_{m-1}, x_m, x_{m+1} all lie between $f(x_{k-1})$ and $f(x_{k+1})$. Suppose $f(x_{k-1}) < x_{m-1}$ and $x_{m+1} < f(x_{k+1})$. (Note that the case where $f(x_{k+1}) < x_{m-1}$ and $x_{m+1} < f(x_{k-1})$ is similar.) Then $f(I_{k-1}) \supset I_{m-1}$ and $f(I_k) \supset I_m$. But $[x_{k-1},1] \supset h(I_{k-1}) \supset g(I_{m-1})$ and $[0,x_{k+1}] \supset h(I_k) \supset g(I_m)$. Thus, as in case 1, $g(I_{m-1}) \supset I_k$ and $g(I_m) \supset I_{k-1}$. This means that x_{k+1} is in $g(I_{m-1})$ and x_{k-1} is in $g(I_m)$. But x_m is a touching point so I_m and I_{m-1} are of the same type so by lemma 7.1 either $f(x_{k+1}) \leq x_m$ or $f(x_{k-1}) \geq x_m$ contrary to the last sentence. Thus we cannot have x_k a down crossing point and x_m a touching point

In any case we reach a contradiction so f and g must permute the fixed points of each type.

Part B

In this section we consider the work of Baxter and Joichi [3]. By dealing just with the basic properties of the permutations $f|H$ and $g|H$ they show that if H is finite and consists only of points which are either up or down crossings, and if the permutations $f|H, f^2|H, \ldots, f^m|H = I|H$ satisfy the A-properties (defined below), then f and g have a common fixed point. Essentially we use only the properties of the permutation $f|H$.

We have shown in part A of this chapter that the permutation $\sigma_f = f|H$ of the elements of H satisfies the following properties which we will call the A-properties. (Let us drop the subscript f in σ_f so $\sigma \equiv \sigma_f$)

(1) σ preserves each of the three types of points

(2) If $\sigma(x_i)$ and $\sigma(x_{i+1})$ are consecutive points with $x_m = \sigma(x_i)$ then whenever $\sigma(x_{i+1}) = x_{m+1}$ the intervals I_i and I_m are of the same type. When $\sigma(x_{i+1}) = x_{m-1}$ the intervals I_i and I_{m-1} are of opposite type.

(3) If $\sigma(x_i)$ and $\sigma(x_{i+1})$ are not consecutive points, then for each $x_v = \sigma(x_j)$ between $\sigma(x_i)$ and $\sigma(x_{i+1})$, $x_j = \sigma^{-1}(x_v) > x_{i+1}$ or $x_j = \sigma^{-1}(x_v) < x_i$ according as I_i is an up- or down-interval respectively.

Suppose now that σ is any permutation on a finite subset of I not necessarily generated by a continuous function. If σ satisfies the A-properties σ^k will clearly satisfy the first A-property but not necessarily the other two.

Definition. Let $H = \{x_1, x_2, \ldots, x_n\}$, $x_i < x_{i+1}$ be a finite subset of I with n odd. Let the intervals $I_i = [x_i, x_{i+1}]$ be specified as alternately down and up with I_1 a down interval. Let the points be classified as before. Then H with the specifications on the intervals and the points will be called a w-set. A permutation of a w-set will be called w-admissible if it satisfies the A-properties relative to H.

Putting our objective in terms of w-sets and w-admissible permutations our aim is to show that if H is a w-set and σ is a permutation of H such that each of $\sigma, \sigma^2, \sigma^3, \ldots, \sigma^{m-1}, \sigma^m = I$ is w-admissible then σ has a fixed point.

To this end the following lemma gives the relation between the period p of x_2 in H and the period of x_1 in H under the permutation σ.

Lemma 7.7. (Baxter and Joichi [3]) Let H be a w-set, and let σ be a permutation of H such that $\sigma^1, \sigma^2, \ldots, \sigma^m = I$ are w-admissible. If the point x_2 has period p, then the point x_2 has period $2p$ or p.

Proof. We show first that $\sigma^k(x_1)$ and $\sigma^k(x_2)$ must be successive points in H. For if not, since I_1 is a down interval, the third A-property requires that $x_j < x_1$ whenever $\sigma^k(x_j)$ is between $\sigma^k(x_1)$ and $\sigma^k(x_2)$. Clearly this is not possible. Let q be the period of x_1, then since $\sigma^q(x_1)$ and $\sigma^q(x_2)$ are successive points and $\sigma^q(x_1) = x_1$ we must have $\sigma^q(x_2) = x_2$. Thus the period of the point x_2 must be a divisor of the period of x_1. Furthermore if p is the period of the point x_2, then $\sigma^p(x_1) = x_1$ or $\sigma^p(x_1) = x_3$. Consider the case where $\sigma^p(x_1) = x_1$. Then the periods of points x_1 and x_2 each divide the other and hence be equal. If $\sigma^p(x_1) = x_3$ and $\sigma^p(x_3) \geq x_5$, then since $\sigma^p(x_2) = x_2$ we would have $\sigma^p(x_1) = x_3$ between $\sigma^p(x_2)$ and $\sigma^p(x_3)$. Thus by the third A-property $x_3 < x_1$. Therefore $\sigma^p(x_3) = x_1$. So $\sigma^{2p}(x_1) = \sigma^p(\sigma^p(x_1)) = \sigma^p(x_3) = x_1$ and the period of x_1 is $2p$.

Lemma 7.8. (Baxter and Joichi [3]) Let H be a w-set and let σ be a permutation of H such that $\sigma^1, \sigma^2, \ldots, \sigma^m = I$ are w-admissible.

Let the points x_1 and x_2 have period $p > 1$ and

$$H' = H \sim \{x_1, \sigma(x_1), \sigma^2(x_1), \ldots, \sigma^{p-1}(x_1), x_2 \sigma(x_2), \ldots, \sigma^{p-1}(x_2)\} .$$

If the elements of H' are $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{n-2p}}\}$ where $x_{i_s} < x_{i_{s+1}}$ and if we classify the points and intervals in H' so that H' becomes a w-set then σ' the restriction of σ to H', and $(\sigma')^k$, $k = 1,2,\ldots$ are w-admissible permutations of H'.

<u>Proof</u>. We shall show that w-admissibility of σ implies that σ' is w-admissible. This will show that the w-admissibility of σ^k implies the w-admissibility of $(\sigma')^k$. We must show that σ' satisfies the A-properties.

(1) The first A-property. Let A_o be the pair $\{x_1, x_2\}$ and A_j be the pair $\{\sigma^j(x_1), \sigma^j(x_2)\}$. Then the union $\bigcup_{i=1}^{p-1} A_i$ is the set of elements belonging to the cycles to which x_1 and x_2 belong. Furthermore we saw in the proof of lemma 7.7 that $\sigma^j(x_1)$ and $\sigma^j(x_2)$ are successive points for all j. By the first A-property, $\sigma^j(x_1)$ is a down point for all j. Thus, if we write H' as in the statement of the lemma, when we classify the points and intervals in H' so as to make H' a w-set, the points retain the classification they had in H. Thus σ' has the first A-property.

(2) The second A-property. We must show that if $\sigma(x_i)$ and $\sigma(x_{i+1})$ are consecutive points with $x_m = \sigma(x_i)$ then whenever $\sigma(x_{i+1}) = x_{m+1}$, the intervals I_i and I_m are of the same type and when $\sigma(x_{i+1}) = x_{m-1}$, the intervals I_i and I_{m-1} are of opposite

type. However this follows directly from the fact that the points $x_m = \sigma(x_i)$ and x_i are of the same type. So I_m and I_i are intervals of the same type while I_{m-1} and I_i are of different types.

(3) The third A-property. If $x_{i_{r+1}} = x_{i_r+1}$ then the interval $[x_{i_r}, x_{i_{r+1}}]$ is an interval determined both by H and H'. Furthermore it retains its original classification. So if $\sigma'(x_r)$ and $\sigma'(x_{r+1})$ are not consecutive and if $\sigma'(x_u)$ is between $\sigma'(x_r)$ and $\sigma'(x_{r+1})$, in other words if $\sigma(x_{i_u})$ is between $\sigma(x_{i_r})$ and $\sigma(x_{i_{r+1}})$, then x_u will satisfy the third A-property with respect to x_r and x_{r+1}.

Suppose now that $x_{i_{r+1}} = x_{i_r+2s+1}$ with $s > o$. Then for each t, $0 \leq t < s$ the pair of points $\{x_{i_r+2t+1}, x_{i_r+2t+2}\}$ is one of the pair A_j and each of the corresponding intervals $[x_{i_r+2t+1}, x_{i_r+2t+2}]$ in H have the same classification, say down. Thus each of the intervals $[x_{i_r+2t+1}, x_{i_r+2t+2}]$, $0 \leq t \leq s$ is an up interval in H_1. So in this case $[x_{i_r}, x_{i_{r+1}}]$ is an up interval in H' because the classification of a point is determined only by the parity of its index which is clearly the same parity as r. For $i_{r+1} - i_r$ is an odd integer so i_r has the same parity as r. Suppose now that $\sigma'(x_u)$ is between $\sigma'(x_r)$ and $\sigma'(x_{r+1})$ so that $\sigma(x_{i_u})$ is between $\sigma(x_{i_r})$ and $\sigma(x_{i_{r+1}})$. To show that the third A property is satisfied, we need to show that $x_u > x_{r+1}$ or equivalently that $x_{i_u} > x_{i_{r+1}}$.
We have that $\{x_{i_r+2t+1}, x_{i_r+2t+2}\}$ is one of the pairs A_j for any t

such that $0 \leq t < s$. Thus the pair $\{\sigma(x_{i_r+2t+1}), \sigma(x_{i_r+2t+2})\}$ is also an A_j so that $\sigma(x_{i_r+2t+1})$ and $\sigma(x_{i_r+2t+2})$ are successive points in H . Thus it follows that $\sigma(x_{i_u})$ must be bewteen $\sigma(x_{i_r+2t})$ and $\sigma(x_{i_r+2t+1})$ for some t, $0 \leq t \leq s$. Therefore $x_{i_u} > x_{i_r} + 2t+1$, for $[x_{i_r+2t}, x_{i_r+2t+1}]$ is an up interval for all $0 \leq t \leq s$ and σ is w-admissible. However x_{i_u} is in H' and $x_{i_u} = x_{i_{r+1}}$. Also x_{i_r} and $x_{i_{r+1}}$ are successive points in H' . Thus $x_{i_u} > x_{i_{r+1}}$ as desired. The case where the interval $[x_{i_r+2t+1}, x_{i_r+2t+2}]$ is an up-interval is similar. Thus the third A-property holds. So σ' is w-admissible on H' as are $(\sigma')^k$.

<u>Lemma 7.9.</u> (Baxter and Joichi [3]) Let H be a w-set, and let σ be a permutation of H such that each of $\sigma, \sigma^2, \ldots, \sigma^m = I$ is w-admissible. If the point x_2 has period $p > 1$ and the point x_1 has period $2p$ then the permutation $\tilde{\sigma}$ obtained from σ by setting

$$\tilde{\sigma}(x_1) = \sigma(x_3)$$

$$\tilde{\sigma}(x_3) = \sigma(x_1)$$

$$\tilde{\sigma}(\sigma^{-1}(x_1)) = x_3$$

$$\tilde{\sigma}(\sigma^{-1}(x_3)) = x_1$$

$$\tilde{\sigma}(x_i) = \sigma(x_i) \quad \text{for all other } x_i$$

satisfies the conditions of lemma 7.8.

Proof. In the proof of lemma 7.7 we showed that for any k, $\sigma^k(x_1)$ and $\sigma^k(x_2)$ are successive points. Since x_1 has period $2p$, $\sigma^p(x_1) = x_3$, and $\sigma^p(x_2) = 2$ it follows that $\sigma^k(x_2) = \sigma^{k+p}(x_2)$ and $\sigma^k(x_3) = \sigma^{k+p}(x_1)$. Hence $\sigma^k(x_2)$ and $\sigma^k(x_3)$ are successive points in H. This implies that $\sigma^k(x_1)$, $\sigma^k(x_2)$ and $\sigma^k(x_3)$ are a successive triple of points in H with $\sigma^k(x_2)$ an up crossing between the other two. Let B_o be the triple $\{x_1 x_2 x_3\}$ and $B_j = \{\sigma^j(x_1), \sigma^j(x_2), \sigma^j(x_3)\}$. Clearly any two of the B_j are disjoint or identical. Write the permutation σ^k in the form

$$\sigma^k : \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{k_1} & x_{k_2} & \dots & x_{k_n} \end{pmatrix} \quad \text{where} \quad x_{k_s} = \sigma^k(x_s)$$

Then each B_j appears in the second row as a consecutive triple either in reverse or proper serial order. If we reverse the order of the elements in a given B_j the resulting permutation will still be w-admissible. In particular if we let $(\sigma^k)^{(j)}$ be the permutation σ^k with the orders of the elements within the triples $B_0, B_1, \dots, B_{j-1}$ reversed in the second row when σ^k is written as above, then $(\sigma^k)^{(j)}$ is w-admissible. Now let $\tilde{\sigma} = \sigma^{(1)}$ then it can be seen that $\tilde{\sigma}^k = (\sigma^k)^{(k)}$ is the $\tilde{\sigma}$ of the lemma. Furthermore the points x_1 and x_2 each have period p with respect to $\tilde{\sigma}$ as is required.

Theorem 7.12. (Baxter and Joichi [3]) Let H be a w-set, and let

σ be a permutation of H such that each of $\sigma, \sigma^2, \sigma^3, \ldots, \sigma^m = I$ is w-admissible. Then σ has a fixed point.

<u>Proof</u>. The proof will be by induction on the number of points in H. The theorem is true for $n = 1$. Now suppose the theorem is true for all k, $1 \leq k < n$. We have by lemma 7.7 that the period of x_1 is either p or $2p$ where p is the period of x_2. In the case where the period of x_1 equals the period for x_2 we have by lemma 7.8 that σ', the restriction of σ to H' has the property that for each $k = 1,2,\ldots,(\sigma')^k$ is a w-admissible permutation. Now the number of elements in H' is less than n the number of elements in H, so σ' must have a fixed point which is also a fixed point of σ.

If the period of the point x_1 is twice the period of x_2 then by lemma 7.9 we arrive at $\tilde{\sigma}$ satisfying the conditions of 7.8. Thus $\tilde{\sigma}$ can be restricted to an H' giving it a fixed point as in the last paragraph. Now this fixed point cannot be an element of any of the cycles to which the points x_1, x_2, or x_3 belong since these points have period $p > 1$. However σ and $\tilde{\sigma}$ agree on the complementary set (which is necessarily non-empty since any two triples B_j are separated in H by at least one up-crossing point). Therefore σ has a fixed point.

<u>Corollary</u>. If H, the set of fixed points of h, is finite with each point being either a down or an up-crossing, and if the permutations $f^2|H$, $f^3|H$, $\ldots$, $f^m|H = I$ are each w-admissible, then f and g have a common fixed point.

Proof. Note that H is a w-set. In part A of this chapter we showed that if f and g commute then $f|H$ satisfies the A-properties. Hence the permutation $\sigma = f|H$ is w-admissible. Now by hypothesis σ^i is w-admissible for each i with $2 \leq i \leq m-1$ and $\sigma^m = I$. Thus by the last theorem σ has a fixed point. So f and g have a common fixed point.

CHAPTER VIII

FULL FUNCTIONS

In previous chapters we have seen positive results in certain special cases, in particular when f and g are polynomials. When we consider the polynomial case we see that the Tchebecheff polynomials each have the interval $[-1,1]$ as their minimal invariant interval. Note that $T_2(x) = 1-2x^2$ maps $[-1,0]$ and $[0,1]$ homeomorphically onto $[-1,1]$, $T_3(x) = 4x^3-3x$ maps each of the intervals $[-1,-\frac{1}{2}]$, $[-\frac{1}{2},\frac{1}{2}]$ and $[\frac{1}{2},1]$ homeomorphically onto $[-1,1]$. Cohen [10] noticed this and he also noted that many pairs of commuting functions with broken line graphs had a similar property. Either I could be subdivided into a number of subintervals on each of which the function is a homeomorphism onto I or it is possible to find a subinterval A of I which is invariant under the function and such that it can be subdivided as above with A replacing I. Cohen called this characteristic fullness. That is, a continuous function f which maps I into itself will be called a <u>full function</u> if there exists a partition $P_f = \{x_0, x_1, \ldots, x_n\}$ of I with $x_0 = 0$, $x_i < x_{i+1}$, $x_n = 1$ and $I_i = [x_i, x_{i+1}]$ such that for each i, $f|I_i$ is a homeomorphism onto I.

This chapter is a composition of work done by Cohen [10], Baxter and Joichi [4], and Folkman [12]. Its purpose is to show that if f is a full function and if g is continuous and commutes with f, then f and g have a common fixed point.

To this end we first prove that a pair of commuting full functions have a common fixed point, a result due to Cohen [10]. Then we consider a special class of full functions which we will call "hat functions". Using Baxter and Joichi's work as a basis we show that any continuous function which commutes with a hat function must be a full function or identically a constant. So by Cohen's result the pair must have a common fixed point.

If it were possible to find for a given full function f, a homeomorphism φ of I onto itself such that $\varphi f \varphi^{-1}$ is a hat function then from what is above, our theorem would be proved. For if g is continuous and commutes with f, then $\varphi g \varphi^{-1}$ is continuous and commutes with $\varphi f \varphi^{-1}$. Thus $\varphi g \varphi^{-1}$ and $\varphi f \varphi^{-1}$ must have a common fixed point and so f and g must have a common fixed point. But it is not always possible to find such a homeomorphism. However Folkman [12] was able to show that for a given full function f there is a function φ which is "almost" a homeomorphism of I onto itself and a hat function $\bar{f}$ such that $\varphi f = \bar{f} \varphi$. (Actually φ is a non-decreasing function of I onto itself which for a certain class of full functions is a homeomorphism.)

If φ is a homeomorphism we use the argument in the last paragraph. Otherwise a short argument depending on the properties of φ gives the desired result.

Our first objective, then, is to prove Cohen's theorem. That is if f and g are a pair of full functions which commute then they have a common fixed point.

At this point it will be useful to make a few remarks to be used later (1) If f and g are functions on the interval $[a,b]$ to itself and φ is a homeomorphism of $[a,b]$ onto $[c,d]$, then $\varphi f \varphi^{-1}$ and $\varphi g \varphi^{-1}$ are functions on $[c,d]$ to itself which commute and have a common fixed point if and only if f and g commute and have a common fixed point. (2) If f and g are commuting functions, then f and $h = gf$ are commuting functions which have a common fixed point iff f and g have.

Lemma 8.1. Let f and g be continuous functions from I to I which commute. If f is monotone then f and g have a common fixed point.

Proof. If f is decreasing, it has a unique fixed point x_o. However we have seen that g maps F, the set of fixed points of f into itself. So x_o must be a fixed point for g. If f is increasing let x_o be in G. Then the sequence $\{f^n(x_o)\}$ is a monotone and bounded sequence of elements of G and thus has a limit which is a common fixed point of f and g. Since G is closed.

Lemma 8.2. (Cohen [10]) If there are continuous commuting functions on I to itself without a common fixed point then there are also onto functions with these properties.

Proof. Let f and g be a pair of functions mapping I into itself and which commute. Let $a_1 = \max[\min f, \min g]$ and $b_1 = \min[\max f, \max g]$.

As we have seen earlier if f and g commute then their graphs must intersect. So their ranges intersect and $a_1 < b_1$. Let f_1 and g_1 be f and g restricted to $[a,b]$. Then f_1 and g_1 take $[a_1,b_1]$ into itself. For, suppose $f_1(x) > b$ for some x in $[a_1,b_1]$. Then let y in $[0,1]$ be such that $g(y) = x$. So $gf(y) = fg(y) = f(x) > b_1$ which implies that $b_1 < \min[\max f, \max g]$. In general let $a_i = \max[\min f_{i-1}, \min g_{i-1}]$, $b_i = \min[\max f_{i-1}, \max g_{i-1}]$, $f_i = f_{i-1}|[a_i,b_i]$, and $g_i = g_{i-1}|[a_i,b_i]$. The set $\{[a_i,b_i] \mid i \geq 1\}$ forms a nested sequence of closed intervals and so has a non-void intersection since I is compact. If the intersection consists of a single point then f and g have a common fixed point. Suppose the intersection is an interval, say $[a,b]$. Clearly the functions $\overline{f} = f|[a,b]$ and $\overline{g} = g|[a,b]$ are onto. Now let φ be a homeomorphism of $[a,b]$ onto I. By the first remark $\varphi\overline{f}\varphi^{-1}$ and $\varphi\overline{g}\varphi^{-1}$ are the required functions.

<u>Definition</u>. A partition P_f is <u>regular</u> if its subintervals are all of the same length. A partition P_g <u>refines</u> P_f <u>uniformly</u> if each P_f interval is the union of the same number of P_g intervals.

<u>Lemma 8.3</u>. (Cohen [10]) If f_1 and g_1 are commuting full functions without a common fixed point, there are functions f and g having the same properties and in addition are such that $f(0) = g(1) = 0$, $f(1) = g(0) = 1$, P_f, P_g and P_{fg} are regular and P_g refines P_f uniformly.

Proof. Since $f_1(0) = g_1(0) = 0$ would be a common fixed point we need only consider the cases where (1) $f_1(0) = 0$, $g_1(0) = 1$ and (2) $f_1(0) = g_1(0) = 1$.

In case (1) $f_1(1) = f_1g_1(0) = g_1f_1(0) = g_1(0) = 1$. Thus $g_1(1)$ must be 0 or else 1 is a common fixed point. In this case let $f_2 = f_1$ and $g_2 = g_1$.

In case (2) $f_1(1) = f_1g_1(0) = g_1f_1(0) = g_1(1)$. Therefore to avoid a common fixed point we must have $f_1(1) = g_1(1) = 0$. In this case let $f_2 = f_1g_1$ and $g_2 = g_1$. Then $f_2(0) = f_1g_1(0) = f_1(1) = 0$, $g_2(0) = g_1(0) = 1$, $f_2(1) = f_1g_1(1) = f_1(0) = 1$, and $g_2(1) = g_1(1) = 0$.

In either case let $f_3 = f_2$ and $g_3 = g_2f_2$. Clearly P_{g_3} refines P_{f_3} uniformly and similarly $P_{f_o g_3}$ refines P_{g_3} uniformly. Now let φ be any order preserving homeomorphism on $[0,1]$ taking $P_{f_3g_3}$ into the corresponding regular partition of I . Then $f = \varphi f_3\varphi^{-1}$ and $g = \varphi g_3\varphi^{-1}$ are the required functions.

Theorem 8.4. (Cohen [10]) If f and g commute and f and g are full functions then they have a common fixed point.

Proof. Assume f_1 and g_1 commute, that they are full functions and that they do not have a common fixed point. Then by lemma 8.3, there are functions f and g having the properties that they are full functions without a common fixed point and in addition $f(0) = g(1) = 0$, $f(1) = g(0) = 1$, P_f, P_g and P_{fg} are regular and P_g refines P_f

uniformly. Let $P_f = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ and $P_g = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}$. Then $P_{gf} = \{0, \frac{1}{mn}, \frac{2}{mn}, \ldots, 1\}$ and m,n are odd. We will denote $f|[\frac{i-1}{n}, \frac{i}{n}]$ by f_i and $g|[\frac{i-1}{m}, \frac{i}{m}]$ by g_i. Let $r = \frac{n+1}{2}$ and $s = \frac{m+1}{2}$. Consider the case where r is odd and s is even. Note that the other cases are similar.

We have that the domain of $f_i g_j$, which we denote by $D(f_i g_j)$, for each i and j is some subinterval of P_{fg} and in particular

$$D(g_1 f_r) = [\frac{r-1}{n}, \frac{r-1}{n} + \frac{1}{mn}]$$

$$D(g_2 f_r) = [\frac{r-1}{n} + \frac{1}{mn}, \frac{r-1}{n} + \frac{2}{mn}]$$

$$\vdots$$

$$D(g_s f_r) = [\frac{r-1}{n} + \frac{s-1}{mn}, \frac{r-1}{n} + \frac{s}{mn}] = [\frac{mn-1}{2mn}, \frac{mn+1}{2mn}] .$$

Similarly we see that

$$D(f_r g_s) = [\frac{s-1}{m} + \frac{r-1}{nm}, \frac{s-1}{m} + \frac{r}{nm}] = [\frac{mn-1}{2mn}, \frac{mn+1}{2mn}] .$$

So $D(f_r g_s) = D(g_s f_r)$. Since g_s is continuous and onto I its graph must intersect the diagonal of $I \times I$ and g_s has a fixed point say z_1. Since $D(g_s) \subset D(f_r)$, z_1 is in $D(f_r)$ and so z_1 is in $D(f_r g_s)$ which is also $D(g_s f_r)$. So $z_2 = f_r(z_1)$ is a fixed point of g_s continuing this process we get a sequence $\{z_p\}$ of fixed points of g_s where $z_{p+1} = f(z_p)$.

Since f_r is monotone the sequence $\{z_p\}$ converges to a point z, which must be a fixed point for f and g. Thus we arrive

at a contradiction. Therefore f and g must have a common fixed point.

Thus we have completed our first step and we are ready to take the next one. We shall give the definition of a hat function and then show that if g is a continuous function which commutes with a hat function f, then g is a constant function or it is also a hat function.

The definition of hat function given here is due to Baxter and Joichi [4].

<u>Definition</u>. A function f is a <u>hat function</u> if for some integer $n \geq 1$, f alternately takes on value 0 and 1 on the points $\frac{i}{n}$ where $0 \leq i \leq n$ and is linear between.

We see that if f is a hat function it is also a full function so that a pair of commuting hat functions must have a common fixed point.

Note that if f is a hat function with n branches, then it has $n+1$ extreme values at the points $\frac{i}{n}$, $0 \leq i \leq n$ and f^k is a hat function with n^k branches and n^k+1 extreme values.

Besides defining hat functions Baxter and Joichi introduce the term "regular" full function. A full function f will be called regular if there is a homeomorphism of I onto itself such that $\bar{f} = \varphi^{-1} f \varphi$ is a hat function.

We are now in a position to describe a few results due to Baxter and Joichi. Although these results are weaker than the theorem

we are aiming for, Baxter and Joichi's theorem came before Folkman's work. The first one is of special interest.

Theorem. (Baxter and Joichi [4]) Suppose f and g commute where f is a regular full function with $n \geq 2$ branches. Then, either g is a constant function $g(x) \equiv c$, where c is a fixed point of f, or g is also a regular full function. In the latter case, there exists homeomorphism φ of I such that $\bar{f} = \varphi^{-1} f\varphi$ and $\bar{g} = \varphi^{-1} g\varphi$ are both hat functions.

This theorem predicts the method that Folkman used in proving his main theorem. It seems likely that it suggested to Folkman what was to be his line of attack. For Folkman shows the existence of a function φ which has a sufficient number of properties of a homeomorphism to be able to carry the burden of the proof.

Their second theorem is the following:

Theorem (Baxter and Joichi [4]) Suppose f and g commute where f is a full function with $n \geq 2$ branches and g is nowhere constant and has only finitely many maxima and minima. Then g is a full function.

This theorem suggests another way to prove that f and g have a common fixed point if f is a full function and g is continuous. Namely, try to show that g must have a finite number of minima and maxima if f is a full function.

Since we will be using a method similar to the idea suggested

by the first of the two theorems mentioned above, we will have to show that if g is continuous and commutes with a hat function f, then f and g have a common fixed point. For this we will need the following lemma.

Lemma 8.5. (Baxter and Joichi [4]) Let f be a hat function with $n \geq 2$ branches and let g commute with f. Then if g is constant on some interval $[a,b]$ contained in I, it is constant everywhere.

Proof. Let $g(x) = c$ for all x in $[a,b]$. Choose i and k so that the interval $[\frac{i}{n^k}, \frac{i+1}{n^k}]$ is contained in $[a,b]$ and consider the function $f^k g = g f^k$. Then $g(I) = gf^k([\frac{i}{n^k}, \frac{i+1}{n^k}]) = f^k g([\frac{i}{n^k}, \frac{i+1}{n^k}]) = f^k(x) = c$.

Baxter and Joichi [4] showed that if f is a hat function with two or more branches and g commutes with f. Then either g is a constant function $g(x) \equiv c$ where c is a fixed point of f, or g is also a hat function. We will be satisfied with

Theorem 8.6. Let f be a hat function with two or more branches and let g commute with f, then either g is a constant function $g(x) \equiv c$ where c is a fixed point of f or g is a full function.

Proof. Evidently the only way a constant function $g(x) \equiv c$ will commute with f is for c to be a fixed point of f. Assume that g is not a constant function. We want to show that (1) g is onto: Let $g(I) = [a,b]$. Then there are integers i and k such that the

interval $[\frac{i}{n^k}, \frac{i+1}{n^k}]$ is contained in $[a,b]$. For such an interval we have

$$g(I) = gf^k(I) = f^k g(I) = f^k[a,b] \subseteq f^k([\frac{i}{n^k}, \frac{i+1}{n^k}]) = I .$$

Thus g is onto.

(2) To show that g is a full function: Since g is onto I, there exist subintervals $[a,b]$ of I such that $g([a,b]) = I$ and $g(J)$ is a proper subset of I whenever J is a proper subinterval of $[a,b]$. (See lemma 4.3.) We will call such intervals minimal - I - intervals for g. Suppose there are m such minimal I intervals for g, say $[a_i,b_i]$ where $i = 1,2,\ldots,m$, with $b_i \leq a_{i+1}$. Now m is in fact finite. For if it was not the sets $A = \{a_i\}$ and $B = \{b_i\}$ would each have an accumulation point in I. Since for any two distinct a_i's there is a b_j between them, any accumulation point of A must be an accumulation point of B. Let c be such a point and let $\{a_{i_k}\}$ be a subsequence of A which has c as a limit. Then $\{b_{i_k}\}$ has c as a limit. Now for each k one of $g(a_{i_k})$ or $g(b_{i_k})$ is 1 and the other is 0. If $g(a_{i_k}) = 1$ call a_{i_k}, d_k otherwise call a_{i_k}, e_k. Do the same for each b_{i_k}. Then the sequences $\{d_k\}$ and $\{e_k\}$ have c as their limit. But $\{g(d_k)\}$ has limit 1 and $\{g(e_k)\}$ has limit 0 contrary to the continuity of g. Hence m must be finite.

Now for any positive integers j and k with $0 \leq j \leq n^k$, the interval $[\frac{j}{n^k}, \frac{j+1}{n^k}]$ will be a minimal - I - interval for f^k. Thus because f^k is monotone on $[\frac{j}{n^k}, \frac{j+1}{n^k}]$, it must contain exactly m minimal - I - intervals for the function gf^k. So I has at least

mn^k minimal - I - intervals for gf^k. The only way that I could have more than mn^k minimal - I - intervals would be if $\frac{j}{n^k}$ was an interior point of some minimal - I - interval for some j. But this cannot happen since $f^k(\frac{j}{n^k} + x) = f^k(\frac{j}{n^k} - x)$ for all x with $0 \leq x \leq \frac{1}{n^k}$. This would mean that for $a_i < \frac{j}{n^k} < b_i$ either $[a_i, \frac{j}{n^k}]$ would be contained in $[\frac{j}{n^k}, b_i]$ or vice versa contrary to $[a_i,b_i]$ being a minimal - I - interval. Thus gf^k will have exactly mn^k minimal - I - intervals. By commutativity this must also be true for the function f^kg. Since each of the m intervals $[a_i,b_i]$ contains at least n^k minimal - I - intervals for f^kg, each must contain exactly n^k such intervals. Because this is true for any k it follows that g must be monotone on each of the intervals $[a_i,b_i]$. For if g was not monotone on $[a_i,b_i]$ there would be a set of three points x_1, x_2 and x_3 with $a_i \leq x_1 < x_2 < x_3 \leq b_i$ such that either $[g(x_1),g(x_2)] = [g(x_2),g(x_3)]$ or $[g(x_2),g(x_1)] = [g(x_2),g(x_3)]$. So for some k and some j, $[\frac{j}{n^k}, \frac{j+1}{n^k}]$ is contained in $[g(x_1),g(x_2)]$ or $[g(x_2),g(x_1)]$ depending on whether $g(x_1) < g(x_2)$ or not. Thus $[a_i,b_i]$ would contain at least n^k+1 minimal - I - intervals which as we have seen is not possible. So g is monotone on $[a_i,b_i]$ and in fact, by the last lemma, strictly monotone. Thus it is clear that $a_1 = 0$ and $b_i = a_{i+1}$ for $i = 1,2,\ldots,m-1$ with $b_m = 1$. Thus g is a full function. This completes the proof.

Note by theorem 8.4, since f is also a full function f and g must have a common fixed point. Recall that Baxter and Joichi showed that if f is a regular full function and g commutes with f then

there is a homeomorphism of I such that $\overline{f} = \varphi^{-1}f\varphi$ and $\overline{g} = \varphi^{-1}g\varphi$ are both hat functions. Hence, it follows that if one of the functions of a commuting pair is a regular full function then the pair has a common fixed point. Although Folkman's lemma is weaker it is strong enough to lead to the proof of a common fixed point for the pair when one of the functions is a full function.

Lemma 8.7. (Folkman [12]) Let f be a full function with $n \geq 2$ branches. There is a continuous non-decreasing function φ mapping I onto I and a hat function $\overline{f}$ with n branches such that $\varphi f = \overline{f}\varphi$. If g commutes with f, there is a continuous function $\overline{g}$ such that $\varphi g = \overline{g}\varphi$ and $\overline{g}$ commutes with $\overline{f}$. Furthermore φ is a homeomorphism if and only if f is regular.

Proof. For each positive integer k, let $0 = t(k,0) < t(k,1) < \dots < t(k,n^k) = 1$ be the points where f^k, assumes the values 0 and 1. Define φ by

$$\varphi(x) = \sup \{\frac{i}{n^k} \mid k > 0 , 0 \leq i \leq n^k, t(k,i) \leq x \} .$$

Clearly φ is non-decreasing with $\varphi(0) = 0$ and $\varphi(1) = 1$. Since $t(k,i) = t(k+1,ni)$ we have $\varphi(t(k,i)) = \frac{i}{n^k}$. So $\varphi(I)$ is dense in I. However φ is monotone, so it can have only jump discontinuities, but the fact that $\varphi(I)$ is dense in I implies that φ has no such discontinuities. So φ must be continuous.

Let $T^k = \{t(k,i) \mid 0 \leq i \leq n^k\}$ and let $T = \bigcup_{k \in I} T^k$. Then T is the set of all points in I which are mapped into 0 or 1 by

some iterate of f . Let x be an element of T so $x = t(k,i)$ for some integers k and i then $f(x) = t(k-1,i)$ so $f(x)$ is a member of T . If x is in T^1 then x is in T^2 so $f(x)$ is in T for any x in T , that is, $f(T)$ is contained in T . If x is an element of T then $x = t(k,i)$ for some k and i so $x = t(k+1,ni)$. Thus x is in $f(T)$. So $f(T)$ contains T and $f(T) = T$. Similarly $f^{-1}(T) = T$.

Furthermore, if $0 \leq x \leq y \leq 1$, then a necessary and sufficient condition for $\varphi(x) = \varphi(y)$ is that the interval $[x,y]$ contain at most one pair from the set T . To see this suppose $x,t \in T$ with $x \neq t$, that t is in (x,y) and that there is no element of T which lies between x and t . Then $x = t(k,i)$ and $t = t(k,i+1)$ for some integers i and k with $0 \leq i \leq n^k$. So $\varphi(x) = \frac{i}{n^k} < \frac{i+1}{n^k} \leq \varphi(y)$ contrary to assumption. A similar argument holds if y is an element of T . So assume there is one element t in $[x,y]$ with $x \neq t \neq y$. Then $\varphi(y) = \varphi(t)$. Let $t = t(k_o,i)$ for some k_o and i . Then $\varphi(x) \geq \frac{i_k - 1}{n^k}$ for every $k \geq k_o$ where i_k is such that $t = t(k,i_k)$. Thus $\varphi(x) \geq \lim_{k \to \infty} \frac{i_k - 1}{n^k} = \frac{i}{n^k} = \varphi(y)$. But $\varphi(x) \leq \varphi(y)$ so $\varphi(x) = \varphi(y)$.

Define $\bar{f}$ by $\bar{f}(x) = \varphi f \varphi^{-1}(x)$. Since $\varphi^{-1}(x)$ might be an interval we must show that $\bar{f}$ is well defined. To do this we will only need to show that $\varphi(x) = \varphi(y)$ implies that $\varphi f(x) = \varphi f(y)$. Suppose this is not the case. Then there are points x and y in I such that at most one point of T is between them and at least two points of T between $f(x)$ and $f(y)$. Now by the intermediate value theorem those points of T which lie between $f(x)$ and $f(y)$

must be the images under f of points which lie between x and y. But $f^{-1}(T) = T$ so we arrive at a contradiction. Therefore $\overline{f}$ is well defined.

Note that $\overline{f}\varphi = \varphi f \varphi^{-1}\varphi = \varphi f$ follows from the previous paragraph. Let Q be a closed subset of I. Then $\overline{f}^{-1}(Q) = \varphi f^{-1}\varphi^{-1}(Q)$. Now $f^{-1}\varphi^{-1}(Q)$ is closed by the continuity of f and φ, and hence is compact. Therefore $\varphi(f^{-1}\varphi^{-1}(Q))$ is compact by continuity of φ and hence is closed so $\overline{f}$ is continuous.

To show that $\overline{f}$ is a hat function with n branches, it is sufficient to show that for each positive integer k and each i, $0 \le i \le n$, $\overline{f}$ maps the points $\frac{i}{n} + \frac{j}{n^{k+1}}$, $0 \le j \le n^k$ monotonically onto the points $\frac{\ell}{n^k}$, $0 \le \ell \le n^k$. This follows from the equation

$$\overline{f}\left(\frac{i}{n} + \frac{j}{n^{k+1}}\right) = \overline{f}(\varphi(t(k+1, n^k i+j)))$$

$$= \varphi f(t(k+1, n^k i+j))$$

$$= \begin{cases} \varphi(t(k,j)) = \dfrac{j}{n^k} & \text{if } f \text{ is increasing on } [t(1,i), t(1,i+1)] \\[2ex] \varphi(t(k,n^k-j)) = \dfrac{n^k-j}{n^k} & \text{if } f \text{ is decreasing on } [t(1,i), t(1,i+1)] \end{cases}$$

Let g be a continuous function which commutes with f. Then $\varphi(x) = \varphi(y)$ implies that $\varphi g(x) = \varphi g(y)$. If not, then for some x and y in I with $x < y$, $\varphi(x) = \varphi(y)$ and $\varphi g(x) \neq \varphi g(y)$. If there is a point t in T such that $x < t < y$, it is the only element of T in $[x,y]$. Furthermore $\varphi(x) = \varphi(t) = \varphi(y)$ but either $\varphi g(x) \neq \varphi g(t)$

or $\varphi g(t) \nparallel \varphi g(y)$. By replacing $[x,y]$ by either $[x,t]$ or $[t,y]$, we may assume that there is no element t in T with $x < t < y$. So f^k is monotone on $[x,y]$ for every k .

Since $\varphi g(x) \nparallel \varphi g(y)$ we have for large enough k an arbitrarily large number of consecutive points from T^k between $g(x)$ and $g(y)$. Thus for any r , there is a sequence of points $x_1,x_2,\ldots,x_m$ with $m \geq r$ between $g(x)$ and $g(y)$ such that f^k alternately assumes the values 0 and 1 on this sequence. Now the sequence $x_1,x_2,\ldots,x_m$ is the image under g of a montone sequence $y_1,y_2,\ldots,y_m$ of points in $[x,y]$. To see this, let a,b be in I . Now $g([a,b])$ is a closed interval containing $g(a)$ and $g(b)$. Hence if c is between $g(a)$ and $g(b)$, then c is the image under g of a point between a and b . Since x_1 and x_m are between $g(x)$ and $g(y)$, there are points y_1 and y_m between x and y such that $g(y_1) = x_1$ and $g(y_m) = x_m$. Now x_2 is between x_1 and x_m , so there is a point y_2 between y_1 and y_m with $g(y_2) = x_2$. Continuing in this fashion, we construct a sequence of points $y_1,y_2,\ldots,y_m$ in $[x,y]$ with $g(y_i) = x_i$ and y_i between y_{i-1} and y_m for $1 < i < m$. This is the required sequence.

The function $f^k g = gf^k$ alternately assumes the values 0 and 1 on the sequence $y_1,y_2,\ldots,y_m$. Since f^k is monotone on $[x,y]$, the sequence $f^k(y_1),f^k(y_2),\ldots,f^k(y_m)$ is monotone and g alternately assumes the values 0 and 1 on this sequence. Since r may be arbitrarily large we get a contradiction to the continuity of g .

The same argument that was used to show that $\bar{f}$ is continuous can be used to show that $\bar{g}$ is continuous.

Now $\bar{f}\,\bar{g}\,\varphi = \bar{f}\varphi g = \varphi fg = \varphi gf = \bar{g}\varphi f = \bar{g}\,\bar{f}\,\varphi$ so $\bar{f}\,\bar{g}\,\varphi = \bar{g}\,\bar{f}\,\varphi$ and because φ is onto $\bar{f}\,\bar{g} = \bar{g}\,\bar{f}$.

The final part of the theorem follows when we observe that if φ is a homeomorphism $\varphi f \varphi^{-1} = \bar{f}$ so f is regular. Conversely if f is regular then the set T is dense in I . This implies that φ is one-to-one and hence a homeomorphism.

Theorem 8.8 (Folkman [12]) Let f be a full function. If g is continuous and commutes with f , then f and g have a common fixed point.

Proof. If f has only one branch, then f is monotone and the conclusion follows from lemma 8.1. If f has more than one branch then by lemma 8.7 there is a hat function $\bar{f}$ such that $\varphi f = \bar{f}\varphi$ and a continuous function $\bar{g}$ such that $\varphi g = \bar{g}\varphi$ and $\bar{f}\,\bar{g} = \bar{g}\,\bar{f}$. By theorem 8.6 either $\bar{g} \equiv c$ where c is a fixed point of $\bar{f}$ or $\bar{g}$ is a full function. Thus by theorem 8.4 $\bar{f}$ and $\bar{g}$ have a common fixed point say x_o. Let $[a,b] = \varphi^{-1}(x_o)$ (that $\varphi^{-1}(x_o)$ is, in fact, an interval is due to the fact that φ is non-decreasing). Then

$$\varphi f([a,b]) = \bar{f}\varphi([a,b]) = \bar{f}(x_o) = x_o$$

so $f([a,b])$ is contained in $[a,b]$ and similarly $g([a,b])$ is contained in $[a,b]$.

If f is not monotone on $[a,b]$, then there is a t in $T^{(1)}$ with $a < t < b$. Now $f(t)$ is in $[a,b]$ and $f(t)$ is in $T^{(1)}$ but $[a,b]$ contains at most one point of $T^{(1)}$ so $f(t) = t$. However $f(t)$ is either 0 or 1 and neither of these points can be in the interior of $[a,b]$. Hence f is monotone in $[a,b]$. Thus by lemma 8.1 f and g have a common fixed point in $[a,b]$.

CHAPTER IX

COUNTER EXAMPLES

So far we have seen that the conjecture holds true for a number of special cases. The fact that the commutativity of functions is such a strong assumption, say in the polynomial case, would lead one to believe that it might be sufficiently strong to ensure a common fixed point for commuting continuous functions in general. However, this is not the case. Huneke [17] and Boyce [7] have each constructed counter examples. Huneke in fact constructed two, one of which is practically identical to Boyce's. The two types of counter examples are sufficiently different to warrant separate consideration.

Part A

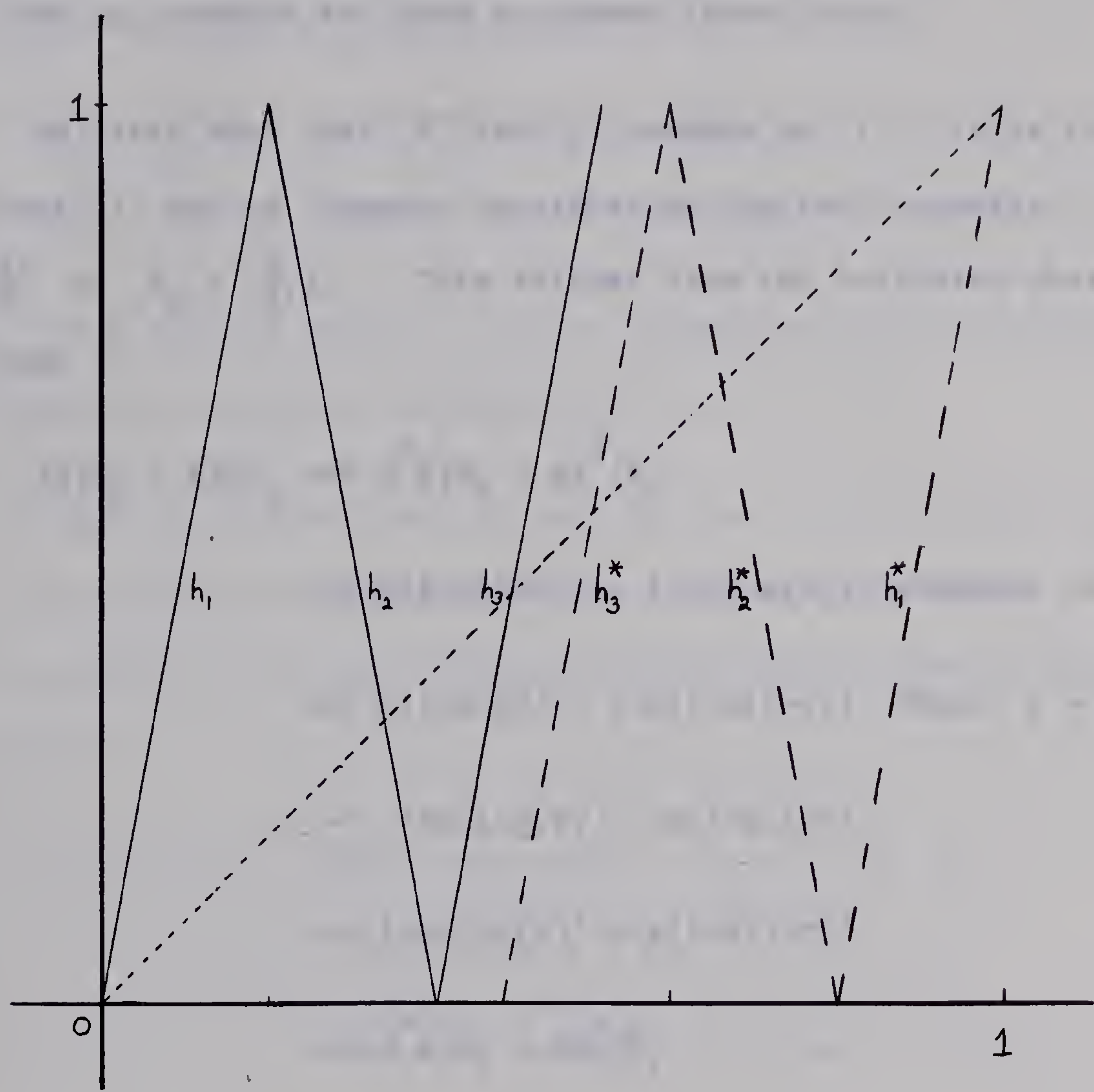
We first consider the second of Huneke's counter examples. To this end we will find it useful to have the following definitions:

(1) Let h be a function from the real line into itself; then by h^* we mean the real function defined by the equation $h^*(x) = 1-h(1-x)$.

(2) Let $s = 3+\sqrt{6}$ and h_1, h_2 and h_3 the real functions defined by the equations $h_1(x) = sx$, $h_2(x) = 2-sx$ and $h_3(x) = sx-2$.

(3) Let $I_1 = [0,\frac{1}{s}]$, $I_2 = [\frac{1}{s},\frac{2}{s}]$, $I_3 = [\frac{2}{s},\frac{3}{s}]$, $I_4 = [\frac{3}{s},1-\frac{2}{s}]$, $I_5 = [1-\frac{2}{s},h_2^{*-1}(\frac{2}{s})]$, $I_6 = [h_2^{*-1}(\frac{2}{s}),1-\frac{1}{s}]$, $I_7 = [1-\frac{1}{s},1]$. A simple calculation shows that $1-\frac{2}{s} < h_2^{*-1}(\frac{2}{s}) < 1-\frac{1}{s}$. So the intervals I_5,

I_6 and I_7 have at most end points in common.



Now suppose there is a continuous function $g : I \to I$ such that

$$g|I_i = h_i \qquad \text{for } i = 1,2,3$$

$$g|I_4 = h_1^{*-1}\ g\ h_3^*$$

$$g|I_5 = h_1^{*-1}\ g\ h_2^*$$

$$g|I_6 = h_2^{*-1}\ g\ h_2^*$$

$$g|I_7 = h_2^{*-1}\ g\ h_1^* \quad .$$

Such a function will be said to have the <u>H-properties</u>. Now let $f = g^*$. Then f and g commute but have no common fixed point.

We first show that f and g commute on I. It is sufficient to show that f and g commute on either of the two intervals $k_1 = [0,\frac{1}{2}]$ or $k_2 = [\frac{1}{2},1]$. This follows from the following chain of implications

$$\begin{aligned}
fg|k_2 = gf|k_2 &\iff g^*g|k_2 = gg^*|k_2 \\
&\iff g(1-g(1-x)) = 1-g(1-g(x)) \quad \text{whenever } x \text{ is in } k_2 \\
&\iff g(1-g(y)) = 1-g(1-g(1-y)) \quad \text{where } y = 1-x \text{ is in } k_1 \\
&\iff -1+g(1-g(y)) = -g(1-g(1-y)) \\
&\iff 1-g(1-g(y)) = g(1-g(1-y)) \\
&\iff g^*g|k_1 = gg^*|k_1 \\
&\iff fg|k_1 = gf|k_1 \quad .
\end{aligned}$$

So f and g commute on k_1 if and only if they commute on k_2.

Now I_i is contained in k_2 for $i = 4,5,6,7$; and $I_3 \cap k_2$ is an interval. Showing that f and g commute on I_i for $i = 4,5,6,7$ is about the same for each i. Consider I_4 as an example.

$$\begin{aligned}
fg|I_4 &= fh_1^{*-1}\, g\, h_3^*|I_4 \\
&= fh_1^{*-1}\, g\, f|I_4 \quad \text{since } f|I_4 = h_3^*|I_4 \\
&= gf \quad \text{since } f|h_1^{*-1}(I) = h_1^* \quad .
\end{aligned}$$

Consider the case of $I_3 \cap k_2$. Note that k_2 is contained in $[1-\frac{3}{s},1]$ so $I_3 \cap k_2$ is contained in $I_3 \cap [1-\frac{3}{s},1]$. On $I_3 \cap [1-\frac{3}{s},1]$, f and g commute since $fg(1-\frac{3}{s}) = gf(1-\frac{3}{s}) = 0$, $fg(\frac{3}{s}) = gf(\frac{3}{s}) = 1$ and both f and g are linear between these points.

Thus we have that f and g commute on k_2 and hence on k_1 . Therefore f and g commute on I .

Now let us show that f and g have no common fixed point. Again it is sufficient to show that they have no common fixed point in either one of k_1 and k_2 . We see this from the following chain of implications

f,g have a common fixed point (C.F.P.) in k_2

$\iff g^*,g$ have a C.F.P. in k_2 . That is

$$g(x) = x = g^*(x) \text{ for some } x \text{ in } k_2$$

$\iff 1-g(x) = 1-x = 1-g^*(x)$

$\iff 1-g(1-y) = y = 1-g^*(1-y)$ where $y = 1-x$ is in k_1

$\iff g^*(y) = y = 1-(1-g(1-(1-y))) = g(y)$

$\iff f(y) = y = g(y)$.

So f,g have a C.F.P. in $k_2 \iff f,g$ have a C.F.P. in k_1 . Then we need only show that f and g have no C.F.P. in k_2 .

Since $f = g^*$ we have that $f|[1-\frac{i}{s},1-\frac{i-1}{s}] = h_i^*|[1-\frac{i}{s},1-\frac{i-1}{s}]$ for $i = 1,2,3$. So the fixed points of f in k_2 will be the fixed

points of the h_i^* , $i = 1,2,3$. Thus we see that the fixed points of f in k_2 are in $I_3 \cap I_4$, I_5 and I_7 .

Now $g(\frac{3}{s}) = 1$ so $\frac{3}{s}$, the point common to I_3 and I_4 , is not a C.F.P. for f and g . $g|I_5 = h_1^{*-1} g h_3^*$ so $g(I_5)$ is contained in $h_1^{*-1}(I) = [1-\frac{1}{s},1]$. Therefore g can have no fixed points in I_5 . Similarly g has no fixed points in I_7 . Thus f and g have no C.F.P.

Thus we must show that there is, in fact, a function g with the H-properties. Here we use an idea suggested by D. W. Boyd. Suppose S is a function space which contains any function having the H-properties. Suppose also that S is complete under the metric d with $d(f,g) \equiv \|f-g\| = \sup_{x \in I} |f(x)-g(x)|$ and that the map $T : S \to S$ is a strict contraction mapping on S where $Tf = f'$ and

$$f'|I_i = h_i \qquad i = 1,2,3$$

$$f'|I_4 = h_1^{*-1} f h_3^*$$

$$f'|I_5 = h_1^{*-1} f h_2^*$$

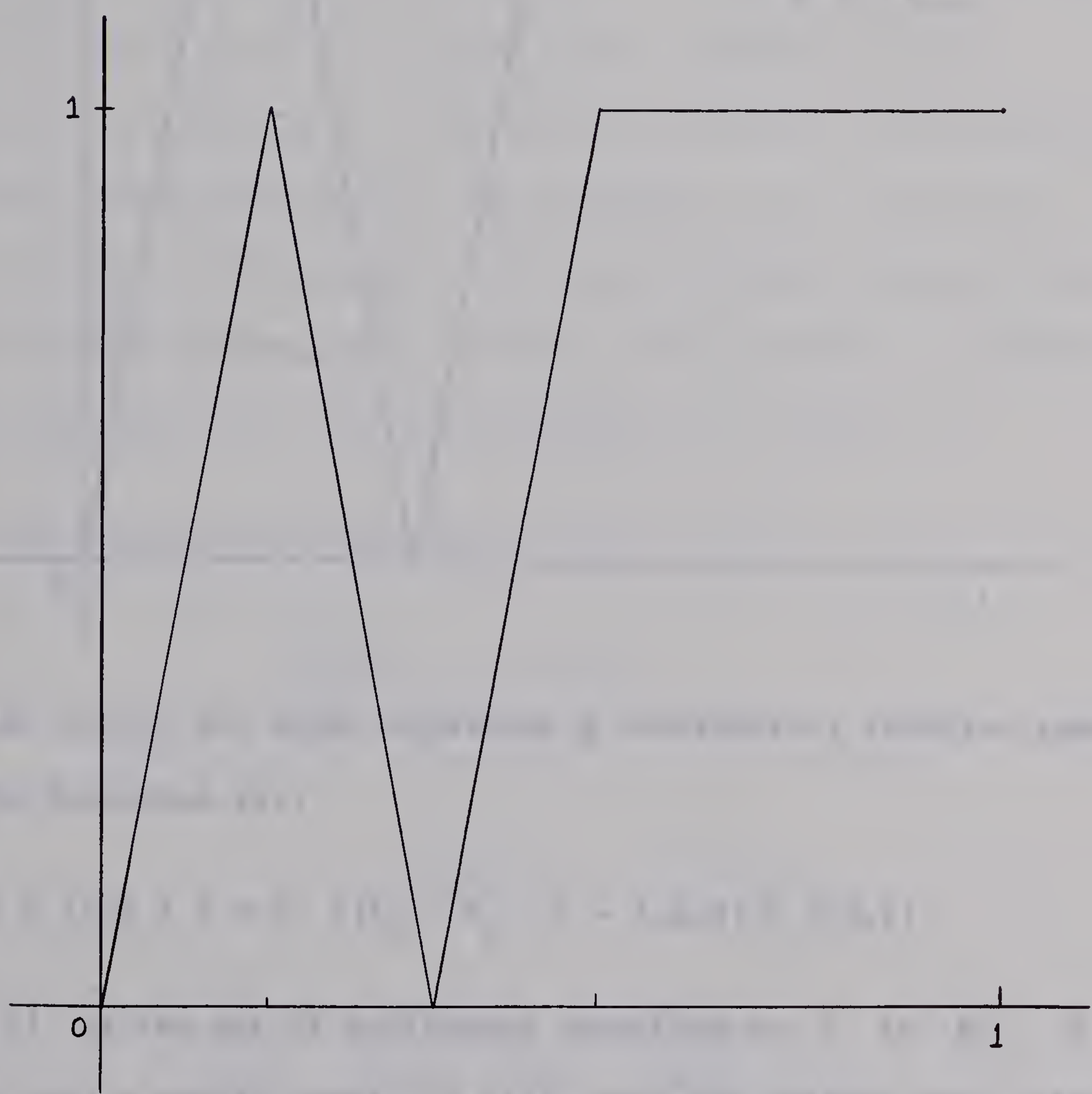
$$f'|I_6 = h_2^{*-1} f h_2^*$$

$$f'|I_7 = h_2^{*-1} f h_1^*$$

By Banach's Contraction Mapping Theorem there is in S a fixed point for T . Call this fixed point g . It is clear that g

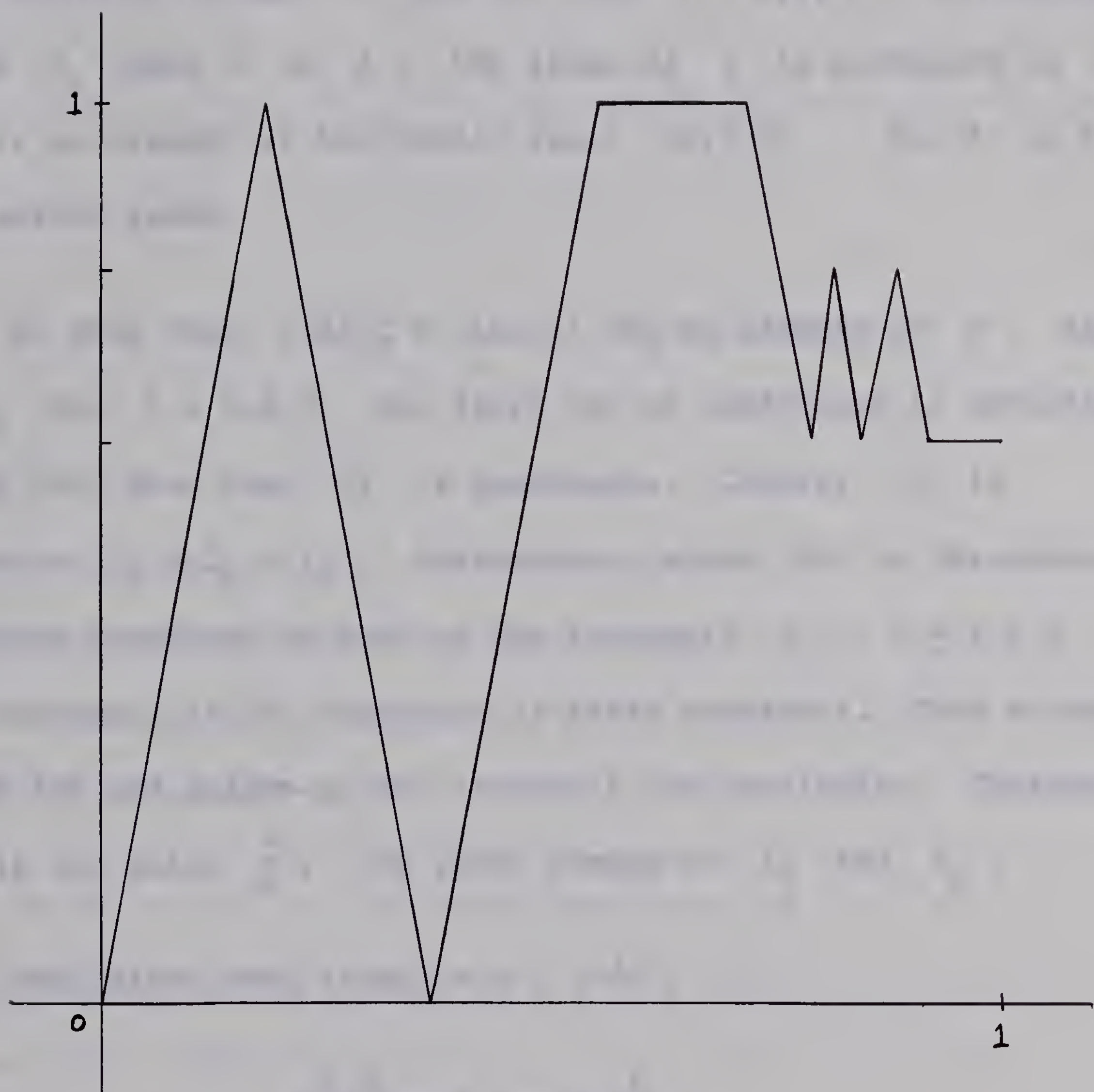
has the H-properties. Furthermore g is unique; that is, g is the only function with the H-properties.

The following two diagrams give a typical f_o and Tf_o



Graph of f_o

Graph of Tf_o



Let us now set about obtaining a satisfactory function space. Consider the following set:

$$S = \{f \mid f : I \to I,\ f|I_i = h_i \quad i = 1,2,3\} \cap C[0,1]$$

where $C[0,1]$ is the set of continuous functions on I to Re . S can be made into a metric space $(S, \|\cdot\|)$ with the uniform norm metric. That is $\|f-g\| = \sup_{x \in I} |f(x)-g(x)|$ defines a metric on S .

Suppose $\{f_n\}$ is a sequence in S which converges to a function f . Then f is the limit of a uniformly convergent sequence

of continuous functions. Therefore f is continuous. Now $f_i|I_j = h_j$ for every positive integer i and for each $j = 1,2,3$. Furthermore, since each f_i maps I to I, the range of f is contained in I. Thus f is an element of the metric space $(S,\|\cdot\|)$. So S is a complete metric space.

To show that $T(S) \subset S$ let f be an element of S. Since $Tf|I_i = h_i$ for $i = 1,2,3,$ the first set of conditions is satisfied so we need only show that Tf is continuous. Clearly Tf is continuous on $I_1 \cup I_2 \cup I_3$. Furthermore, since Tf is the composition of continuous functions on each of the intervals I_i, $4 \leq i \leq 7$, it must be continuous in the interiors of these intervals. Thus we need only check the end points of the intervals for continuity. Consider for example the point $\frac{3}{s}$, the point common to I_3 and I_4.

Note first that since $s = 3 + \sqrt{6}$,

$$\begin{aligned} h_3^*(\tfrac{3}{s}) &= 3 - s(1 - \tfrac{3}{s}) \\ &= 6 - s \\ &= \tfrac{3}{s} . \qquad * \end{aligned}$$

We have

$$\lim_{x \to \frac{3}{s}-} Tf(x) = \lim_{x \to \frac{3}{s}-} (sx-2) = 1$$

and

of continuous functions. [illegible]

[illegible]

Thus $\mathfrak{A}$ is an element of [illegible]

complete metric space.

To show that [illegible]

[illegible]

[illegible]

continuous on [illegible]

of continuous functions [illegible]

must be continuous [illegible]

only check the continuity [illegible]

the numbers the point [illegible]

Here (2.13) is satisfied [illegible]

$$[illegible]$$

we have

$$[illegible]$$

and

$$\lim_{x \to \frac{3}{s}+} T\, f(x) = \lim_{x \to \frac{3}{s}+} h_1^{*-1}\, f\, h_3^*(x)$$

$$= h_1^{*-1}\, f\, h_3^*(\tfrac{3}{s}) \quad \text{since } h_1^{*-1},\ f,\ h_3^* \text{ are continuous}$$

$$= h_1^{*-1}\, f(\tfrac{3}{s}) \quad \text{by } *$$

$$= h_1^{*-1}\,(1) \quad \text{since } f \text{ is in } S\ .$$

$$= 1\ .$$

Thus Tf is continuous at $\frac{3}{s}$. We see that the value of $3 + \sqrt{6}$ is necessary to ensure continuity at the end points of the intervals I_i, $4 \leq i \leq 7$. The other end points can be treated in a similar manner. Thus $T(S)$ is contained in S.

To show that T is a contraction mapping on S. Let f and g be elements of S. Then for x an element of any of I_1, I_2 and I_3 we have $|Tf(x)-Tg(x)| = 0$. If x is in I_4 then

$$|Tf(x)-Tg(x)| = |h_1^{*-1}\, f\, h_3^*(x) - h_1^{*-1}\, g\, h_3^*(x)|$$

$$= \left| \frac{fh_3^*(x)-1}{s} - \frac{gh_3^*(x)-1}{s} \right|$$

$$= \frac{1}{s}\, |fh_3^*(x) - gh_3^*(x)|$$

$$\leq \frac{1}{s}\, \|f-g\|\ .$$

A similar argument will work for x in I_5, I_6, or I_7. Thus for all x in I we have $|Tf(x)-Tg(x)| \leq \frac{1}{s}\, \|f-g\|$. Hence

$$\|Tf-Tg\| \leq \frac{1}{s} \|f-g\| \quad .$$

Therefore since $s > 1$, T is a contraction map on S. By Banach's Contraction Mapping Tneorem, T must have a fixed point g in S. As we have seen g is the unique function with the H-properties. Thus $f = g^*$ and g exist, commute on I, and have no common fixed points.

Not only does the function g exist but g' exists a.e. on I. Furthermore $|g'| = s$ a.e. on I. To show this let $O_i = \text{int } I_i$ for $1 \leq i \leq 7$, $O = \text{int } I = (0,1)$ and $B = \bigcup_{i=4}^{7} O_i$. Let $\varphi : B \to O$ be defined by

$$\begin{aligned} \varphi \mid O_4 &= h_3^* \\ \varphi \mid O_5 \cup O_6 &= h_2^* \\ \varphi \mid O_7 &= h_1^* \quad . \end{aligned} \tag{i}$$

Then,

$$Tf \mid O_4 \cup O_5 = h_1^{*-1} \, f \, \varphi$$

and

$$Tf \mid O_6 \cup O_7 = h_2^{*-1} \, f \, \varphi$$

We use open intervals since the end points of the I_i $4 \leq i \leq 7$, the points at which the definition of Tf changes, are not in B. So Tf has a derivative at x whenever f has a derivative at $\varphi(x)$. Notice that

$$|(Tf)'(x)| = \left| \frac{1}{|s|} \cdot f'(\varphi(x)) \cdot |s| \right|$$

$$= |f'(\varphi(x))| \qquad (**)$$

whenever x is in B.

Let $A_o = \bigcup_{i=1}^{3} O_i$. By definition $|g'(x)| = s$ for x in A_o. Let $A_1 = \varphi^{-1}(A_o)$, then if x is in A_1, $\varphi(x)$ is in A_o so $g'(\varphi(x)) = \pm s$ and hence by $(**)$

$$(Tg)'(x) = \pm g'(\varphi(x)) = \pm s \quad .$$

But $Tg = g$ so $g'(x) = \pm s$ for x in A_1.

In general if $g'(x) = \pm s$ for x in E with E an open subset of $(0,1)$, then as above $g'(x) = \pm s$ for x in $\varphi^{-1}(E)$. Thus, let

$$A_2 = \varphi^{-1}(A_1)$$

$$A_3 = \varphi^{-1}(A_2)$$

$$A_n = \varphi^{-1}(A_{n-1})$$

and note that A_o and A_1 are disjoint. Suppose that $A_o, A_1, \ldots, A_n$ are disjoint. Then $\varphi^{-1}(A_o), \varphi^{-1}(A_2), \ldots, \varphi^{-1}(A_n)$ are disjoint, that is, $A_1\ A_2 \ldots A_{n+1}$ are disjoint. Now A_o is outside the domain of definition of φ so $A_o, A_1, \ldots, A_{n+1}$ are disjoint. Therefore $A_o\ A_1 \ldots A_n \ldots$ are disjoint. Furthermore we have that for x in $\bigcup_{n=o}^{\infty} A_n$, $g'(x)$ exists and equals $\pm s$.

To find the measure of the set above, let E be an open set contained in $R_i = h_1(0)$ where $i = 1,2,3$. Then, since h_i^* has slope $\pm s$ depending on whether $i = 1, 3$ or 2, the Lebesgue measure of $h_i^{*-1}(E)$ is $\frac{1}{s}$ times the measure of E. That is

$$m(h_i^{*-1}(E)) = \frac{1}{s} \cdot m(E) \quad .$$

Thus from the definition of φ we have that for any open set in I

$$m(\varphi^{-1}(E)) = \frac{1}{s}\, m(R_1 \cap E) + \frac{1}{s}\, m(R_2 \cap E) + \frac{1}{s}\, m(R_3' \cap E)$$

where $R_3' = h_3^*(O_4)$. Now $R_1 = R_2 = O$ and $R_3' = B \cup T$ with T finite, so for any open set E contained in I

$$m(\varphi^{-1}(E)) = \frac{2}{s}\, m(E) + \frac{1}{s}\, m(E \cap B)$$

$$= \frac{3}{s}\, m(E \cap B) + \frac{2}{s}\, (E \cap A_o) \qquad \text{(ii)}$$

If $E = \bigcup_{n=o}^{\infty} A_n$ then

$$\varphi^{-1}(E) = \bigcup_{n=o}^{\infty} \varphi^{-1}(A_n)$$

$$= \bigcup_{n=1}^{\infty} A_n$$

$$= E \sim A_o$$

so

$$m(\varphi^{-1}(E)) = m(E) - m(A_o) \quad .$$

But from (ii) we have that

$$m(\varphi^{-1}(E)) = \frac{2}{s} m(A_o) + \frac{3}{s} m(E \sim A_o) \quad .$$

Equating the right hand side of the last two equations we get,

$$m(E)-m(A_o) = \frac{2}{s} m(A_o) + \frac{3}{s} m(E) - \frac{3}{s} m(A_o) \quad .$$

Thus

$$m(E) = \frac{1 - \frac{1}{s}}{1 - \frac{3}{s}} m(A_o) = \frac{s-1}{s-3} \cdot \frac{3}{s} = 1 \quad .$$

Hence $g'(x) = \pm s$ a.e. on I .

The above counter example is one of a class of such functions which Huneke describes. Essentially he defines a continuum of maps E_b for $0 < b \leq \frac{1}{2}$ from C_1 , the set of continuous functions of I to itself with $f(1) = 1$, to the set $C[0,1]$. For a given b in $[0,\frac{1}{2})$, Huneke shows that any two functions say f and g in the range of E_b have the properties that f and g^* commute and have no common fixed point. Let us consider the maps E_b .

Let $s = s(b) = \dfrac{3 - 2b + \sqrt{6-4b}}{1 - 2b}$ and define the linear functions $h_{b,i}$ as follows

$$h_{b,1}(x) = sx - sb + b$$

$$h_{b,2}(x) = 2 - sx + sb - b$$

$$h_{b,3}(x) = -2 + sx - sb + b \ .$$

Let

$$I_{b,0} = [0,b] \ , \quad I_{b,1} = [b, \frac{1-b+sb}{s}] \ , \quad I_{b,2} = [\frac{1-b+sb}{s}, \frac{2-b+sb}{s}]$$

$$I_{b,3} = [\frac{2-b+sb}{s}, \frac{3-2b+sb}{s}] \ , \quad I_{b,4} = [\frac{3-2b+sb}{s}, 1 - \frac{2-b+sb}{s}]$$

$$I_{b,5} = [1 - \frac{2-b+sb}{s}, h_{b,2}^{*-1}(\frac{2-b+sb}{s})], \ I_{b,6} = [h_{b,2}^{*-1}(\frac{2-b+sb}{s}), 1 - \frac{1-b+sb}{s}]$$

$$I_{b,7} = [1 - \frac{1-b+sb}{s}, b] \ , \quad I_{b,8} = [1-b, 1]$$

and let for a given f in C_1 , $E_b f = g_b$ where

$$\begin{aligned}
g_b | I_{b,0} &= b \cdot f(x/b) \\
g_b | I_{b,i} &= h_{b,i} \quad \text{for } i = 1,2,3 \\
g_b | I_{b,4} &= h_1^{*-1} \ g_b \ h_3^* \\
g_b | I_{b,5} &= h_1^{*-1} \ g_b \ h_2^* \qquad \text{(iv)} \\
g_b | I_{b,6} &= h_2^{*-1} \ g_b \ h_2^* \\
g_b | I_{b,7} &= h_2^{*-1} \ g_b \ h_1^* \\
g_b | I_{b,8} &= \text{fixed point of } h_{b,2}^* \ .
\end{aligned}$$

Notice that $s(0)$ is equal to the s of the above counter example, that for $b = 0$, $I_{b,0} = I_{b,8} = \varphi$, $I_{b,i} = I_i$ $1 \leq i \leq 7$ and that $h_{b,i} = h_i$. So the function $g_o = E_o(f)$ is the image of each f in C_1 and is also the fixed point of T on S .

As it was shown for the special case $(b = 0)$, it can be

shown that equations (iv) define a continuous function from I to itself for each f in C_1. Also for a given f in C_1, $E_b(f)$ and $[E_b(f)]^*$ commute and have no common fixed point. What is more, it can be shown that for f and g in C_1, $E_b(f)$ and $[E_b(g)]^*$ are continuous, commute and have no common fixed points.

Part B A Second Counter Example.

Working independently of Huneke, W. M. Boyce [7] constructed a counter example which is essentially the same as one of the counter examples constructed by Huneke. The idea for this second example is quite different from the idea of the first one. Both Huneke and Boyce thought of trying to obtain the functions f and g as limits of a pair of sequences $\{f_n\}$ and $\{g_n\}$ of continuous functions.

It is clear that if we attempt to have f_n and g_n commute for each n, then we cannot obtain a limiting pair of continuous functions which do not have a common fixed point unless all but a finite number of the pairs $\{f_n, g_n\}$ have these properties. However, it is possible to get around this difficulty by requiring instead, that $f_{n-1}\, g_n = g_{n-1}\, f_n$ for each n. Then we must make sure that the fixed points of f_n are bounded away from the fixed points of g_n for sufficiently large n.

To ensure that the functions f and g are continuous we want each f_m and g_m to be continuous and the sequences $\{f_n\}$ and $\{g_n\}$ to converge uniformly to f and g respectively.

Now the functions f_n and g_n are each defined inductively from both f_{n-1} and g_{n-1}. In fact f_{n-1} and g_{n-1} determine a unique f_n. Also a unique g_n is determined by f_{n-1} and g_{n-1}. Thus the choice of the initial pair of functions is crucial. This pair must be chosen so that the conditions in the above paragraphs can be satisfied.

Following Boyce's work we can think of a finite set of "stable points" S_i as being associated with the pair of functions $\{f_i, g_i\}$. A point x is in S_i if and only if $f_i(x) = f_{i+k}(x)$ and $g_i(x) = g_{i+k}(x)$ for each positive integer k. So each set S_i is contained in S_j whenever $j \geq i$. We want $\bigcup_{i=o}^{\infty} S_i$ to be dense in I so that the limit functions will be independent of the values of f_i and g_i off of S_i.

One way of insuring that $\bigcup_{i=o}^{\infty} S_i$ is dense in I to require that for each i, if x,y are consecutive elements in S_i with $x < y$, then there are at least two points in S_{i+1} between x and y which partition the interval $[x,y]$ into intervals of equal length. Note that if $M_i = \max\{|x-y| \mid x,y$ are consecutive elements of $S_i\}$ then $M_{i+1} \leq \frac{1}{3} M_i$.

Another requirement which must be satisfied by $\{f_n\}$ and $\{g_n\}$, that of uniform convergence, can be dealt with by considering the subset $S_{i-1} \times S_i$ of $I \times I$. Let x,y in S_{i-1} and x',y' in S_i be consecutive pairs in S_{i-1} and S_i respectively with $x < y$ and $x' < y'$. Then the four ordered pairs in $S_{i-1} \times S_i$ determined by these four points are the vertices of a rectangle in $I \times I$. Suppose for every i in N and for each consecutive pair x,y in S_i,

there is only one consecutive pair in S_{i-1} such that $f_i \mid [x,y]$ is contained in the rectangle determined by these points. Suppose also that whenever the graph of f_i (restricted to the appropriate interval) is contained in one such rectangle so is the graph f_{i+1}. Then $\{f_i\}$ converges uniformly to f, since the height of the rectangle tends to zero as i grows without bound.

By requiring that the graph of f_{i+1} lie within the same rectangle, determined by $S_{i-1} \times S_i$, as does f_i, we cause the fixed points of f_{i+1} and f to be located near the fixed points of f_i. Thus, by choosing a pair of functions f_1 and g_1 with fixed points "far apart" we keep the fixed points of f_i and g_i bounded away from each other.

Let us now define the functions f_n and g_n, $n = 1,2,3,\ldots$. In order to do this we will need to define a few auxiliary terms

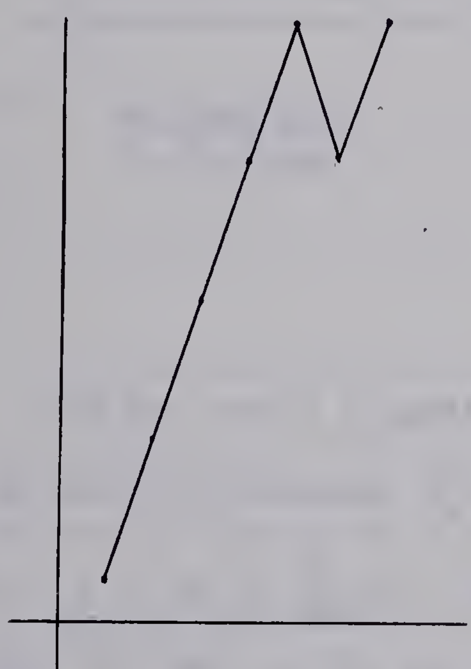
(1) Let S_i be a finite subset of I indexed by the non-negative integer i. Then a closed interval J contained in I will be called an _(i)-interval_ if its end points are in S_i but none of its interior points are.

(2) For a closed interval A the subset T of A is the _k-set_ of A if it has k elements, contains the end points of A, and divides A into $k-1$ subintervals of equal length.

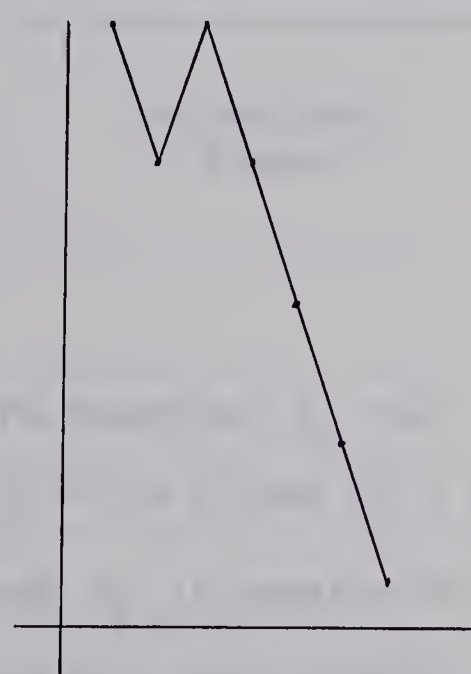
(3) Let A and B be closed intervals, let T be the $2k+2$ set of A and U be the $2k$ set of B for $k \geq 2$. Let the points of T be $t_1, t_2, \ldots, t_{2k+2}$ in either ascending or descending order and the

the points of U be $u_1, u_2, \ldots, u_{2k}$ in either ascending or descending order. Let f be a continuous function defined as follows: $f(t_1) = u_1$, $f(t_2) = u_2$, $f(t_i) = u_{i-2}$ for $i \geq 3$, with f defined to be linear between points of T. Then f is a (2k+2)-hook function from A onto B. The order of f is $2k+2$ and the type of f is either maximum hook or minimum hook according to whether u_1 is the maximum or minimum element of B.

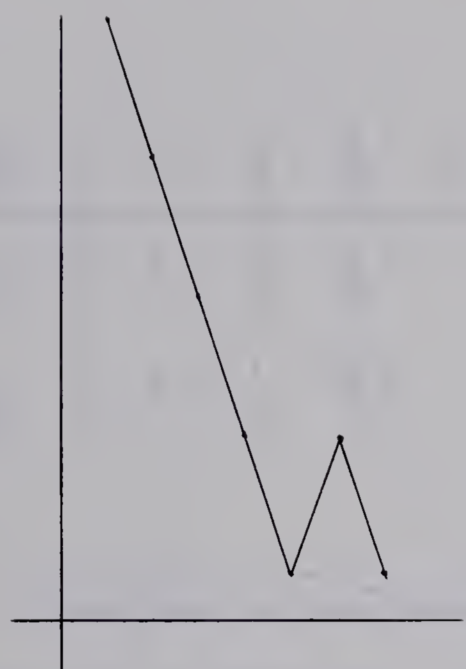
(4) A function f on an interval $A = [x,y]$ will be said to be rising on A if $f(y) > f(x)$, otherwise falling on A.



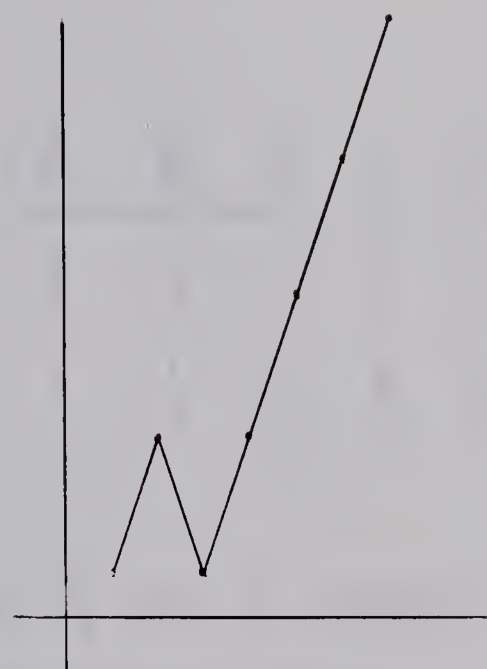
Maximum Hook
Rising



Maximum Hook
Falling



Minimum Hook
Falling



Minimum Hook
Rising

We are now in a position to define the functions f_n and g_n. We start by choosing $S_o = \{0,1\}$ and S_1 as the (4)-set of I, that is, $S_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Then $M_1 = \frac{1}{3}$ and S_o is contained in S_1. Let f_1 and g_1 be the two hat functions with three branches determined by $f_1(0) = 1$ and $g_1(0) = 0$. Then f_1 and g_1 take S_1 onto S_o and are linear between points of S_1. Now f_1 and g_1 have a common fixed point. However f_2 and g_2 will not have common fixed points nor will any of the succeeding pairs of functions f_i and g_i.

Let S_2 consist of the union of the (4)-sets for the first and third intervals and the (6)-set for the middle (1)-interval. Thus $S_o \subset S_1 \subset S_2$. Since f_2 and g_2 are linear between points of S_2 we have that f_2 and g_2 are completely determined by the following table

S_2	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{6}{15}$	$\frac{7}{15}$	$\frac{8}{15}$	$\frac{9}{15}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1
f_2	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
g_2	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1

Note that on the middle (1)-interval f_1 and g_1 are hook functions. It is clear that $f_1 g_2 = g_1 f_2$. We see also that f_2 restricted to the middle (1)-interval is a minimum hook function rising and g_2 restricted to the same interval is a minimum hook falling.

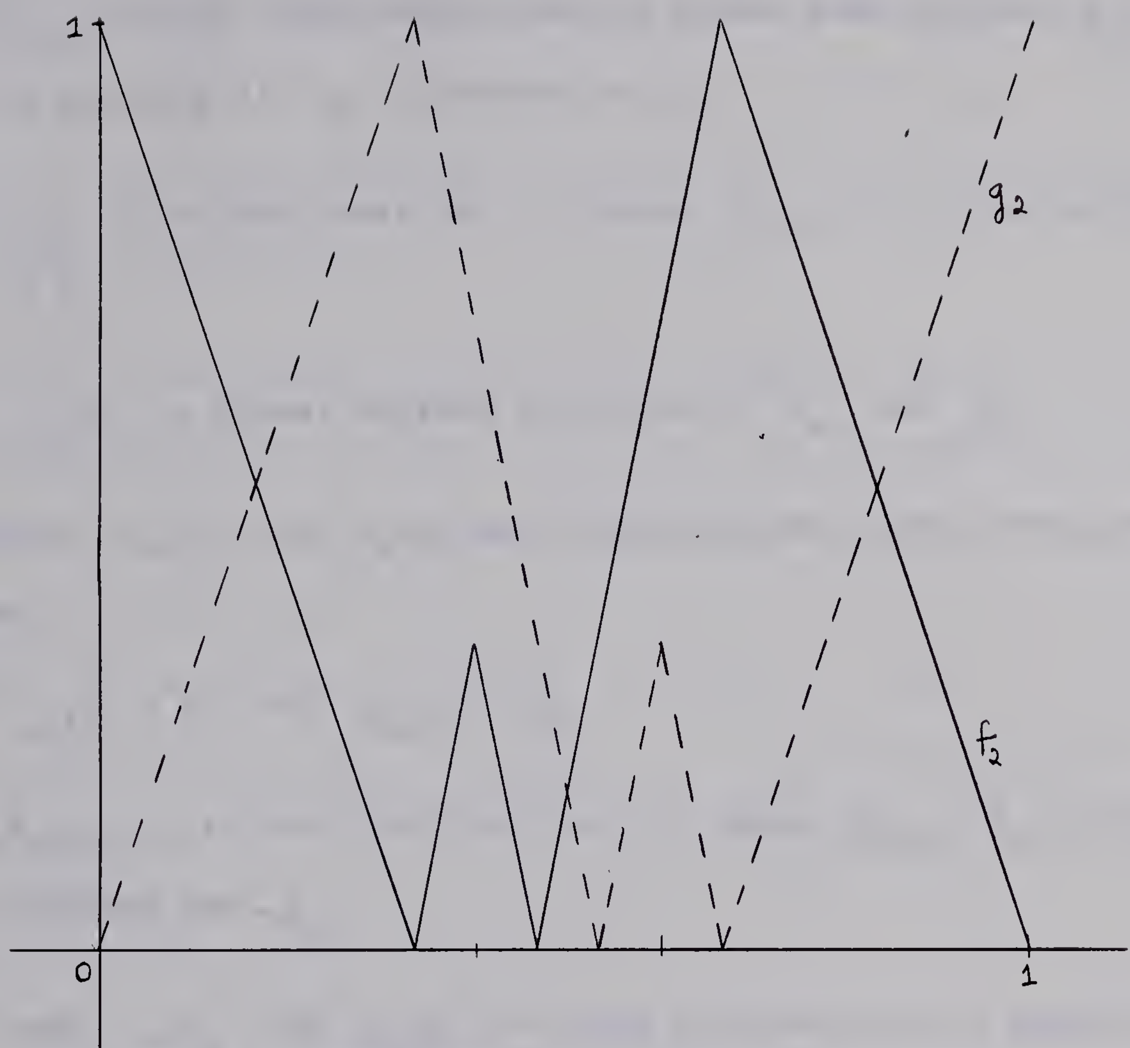
Now we will define for each n the functions f_{n+1} and g_{n+1} assuming f_n, g_n and S_n have been defined. The set S_{n+1} will also be defined along with f_{n+1} and g_{n+1}. Suppose f_n, g_n and S_n have been defined. Let J be an (n)-interval then $J_f = f(J)$ and $J_g = g(J)$ are $n-1$ intervals and $J_h = (J_f)_g = (J_g)_f$ is an (n-2) interval.

[illegible]

[illegible]

[illegible]

[illegible]



Graphs of g_2 and f_2

(I) If $g_n|J_f$ and $f_n|J_g$ are both linear define

(a) $f_{n+1}|J = f_n$ and $g_{n+1}|J = g_n$

(b) $S_{n+1} \cap J$ to be the (2k)-set for J where $S_{n-1} \cap J_k$ is the (2k)-set of J_h

(II) If one of $g_n|J_f$ and $f_n|J_g$ is linear and the other is a hook function (say $f_n|J_g$ is linear) define

(a) $f_{n+1}|J = f_n$

(b) g_{n+1} is the $(2k+2)$-hook function of the same type as $g_n|J_f$ if and only if f_n is rising on J_g .

(c) S_{n+1} is a $(2k+2)$-set for J where $S_{n-1} \cap J_h$ is a $(2k)$-set of J_h .

<u>Note</u>: if $g_n|J_f$ is linear replace the roles of f_n and g_n .

(III) If both $f_n|J_g$ and $g_n|J_f$ are hook functions of the same type define

(a) $f_{n+1}|J = f_n$ and $g_{n+1}|J = g_n$

(b) $S_{n+1} \cap J$ is the $(2k+2)$-set for J where $S_{n-1} \cap J_h$ is the $(2k)$-set for J_h

(IV) If both $f_n|J_g$ and $g_n|J_f$ are hook functions but of opposite type define

(a) $f_n|J$ to be hook function of the same type as $f_n|J_g$ if g_n is rising on J_f

(b) $g_{n+1}|J$ to be a hook function of the same type as $g_n|J_f$ if f_n is rising on J_g .

(c) $S_{n+1} \cap J$ is the $(2k+4)$-set of J where $S_{n-1} \cap J_h$ is the $(2k)$-set of J_h .

Since on each (1)-interval both f_2 and g_2 are either linear or hook functions we have that for each (2)-interval f_3 is either linear or a hook function. Similarly g_3 is either linear or a hook function

(ii) $f_{n,m}$ [illegible]

[illegible]

(iii) $f_{n,m}$ [illegible]

[illegible]

(VIII) If $f_{n,i}$ [illegible]

(VIII) If each F_n [illegible] such that

[illegible]

(a) $\sigma_{n,i}$ [illegible] and $\sigma_{n,i}$ [illegible]

(b) $\sigma_{n,i,j}$ [illegible]

[illegible]

(IX) If [illegible] $f_{n,i}$ [illegible]

[illegible]

(c) $f_{n,i}$ [illegible]

[illegible]

(d) $f_{n,i}$ [illegible]

[illegible]

(e) $f_{n,i}$ [illegible]

[illegible]

[illegible] functions [illegible] a [illegible] function [illegible]

on each (2)-interval. So by the inductive definition we see that on each (n-1)-interval f_n is either linear or a hook function. The same thing is true for g_n. Thus through the inductive definition, f_n and g_n are uniquely determined by f_i, g_i and S_i $i = 1,2$.

It can be verified that for each $n, f_n g_{n+1} = g_n f_{n+1}$, f_n and g_n are continuous, and that $\{f_n\}$ and $\{g_n\}$ converge uniformly to the functions f and g respectively. We shall show that f and g can have no common fixed point. Note that if J is an (n)-interval and $f_n(J) = J_f$ is the (n-1)-interval onto which f maps J, then $f_{n+m}(J) = J_f$ for all positive integers m. A similar statement holds for functions g_n. Furthermore $f(x) = f_2(x)$ and $g(x) = g_2(x)$ for all x in s_2.

Thus it is clear from the graphs of f_2 and g_2 that f can cross the diagonal only in the third, fourth, seventh, eighth and ninth (2)-intervals and that g can cross the diagonal only in the first, fifth, and eleventh (2)-intervals. If f has no fixed points in the fourth (2)-interval then all the fixed points of f are bounded away from all fixed points of g. Clearly the only possible fixed point for f in the fourth (2)-interval would be $\frac{1}{3}$ but $f(\frac{1}{3}) = f_2(\frac{1}{3}) = 0$. Thus f and g have no common fixed points.

Recall that the first counter example considered had a derivative $\pm s = \pm (3 + \sqrt{6})$ a.e. on I. We will show that for this second counter example there is a nested sequence of intervals J_n such that on each J_n, f_n has a derivative with an absolute value of

at least $\frac{5^{n-1}}{3^{n-2}}$ where the slope is defined. Since the absolute value of the derivative of f is at least as large as the absolute value of the derivative of f_n, because $f|S_n = f_n|S_n$, the slope of f must be unbounded.

Let J_n' be the (n)-interval containing 0 and J_n'' be the (n)-interval containing 1. Let J_3 be the (3)-interval containing $\frac{4}{9}$ and for $n > 3$ let J_{n+1} be the (n)-interval contained in J_n such that $g(J_{n+1}) = J_n$.

Note that for $n = 2$, $g_n|J'_{n-1}$, $f_n|J'_{n-1}$, $g_n|J''_{n-1}$ and $f_n|J''_{n-1}$ are linear. Thus by (I) of the definition f_{n+1} and g_{n+1} they are all linear for $n+1$ if they are all linear for n. Thus for each n, $g_n|J_n'$ is linear.

Now it is clear that $f(J_3) = J_2'$ and $f(J_n) = J'_{n-1}$. Thus $f_3|g(J_3) \equiv f_3|J_2$ and $f_n|g(J_n) \equiv f_n|J_{n-1}$ since $g(J_n) = J_{n-1}$. So $f_n|J_n$ a hook function implies by II that $f_{n+2}|J_{n+1}$ is a hook function. However $f_3|J_2$ is a hook function. Thus by induction $f_{n+1}|J_n$ is a hook function for each $n \geq 3$. Now by II the order of $f_{n+1}|J_n$ is the same as the order of $f_n|J_{n-1}$ so that $f_{n+1}|J_n$ is a (6)-hook function for each n.

By changing the linear function $f_n|J_n$ to a (6)-hook function the derivative is changed from an absolute value of s to one of absolute value $\frac{5s}{3}$. $f_2|J_1$ has a derivative $|s_2| = 5$ so $|s_n| = \frac{5^{n-1}}{3^{n-2}}$. Thus the derivative of $f|J_n$ has an absolute value

greater than $\frac{5^{n-1}}{3^{n-2}}$ wherever it exists. Thus, if $x_o \in \bigcap_{k=1}^{\infty} J_n$,

then either $|f'(x_o)| = \infty$ or else $f'(x_o)$ does not exist.

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